Monodromy at infinity of polynomial maps and Newton polyhedra*

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Abstract

By introducing motivic Milnor fibers at infinity of polynomial maps, we propose some methods for the study of nilpotent parts of monodromies at infinity. The numbers of Jordan blocks in the monodromy at infinity will be described by the Newton polyhedron at infinity of the polynomial.

1 Introduction

The aim of this paper is to study the nilpotent parts of monodromies at infinity of polynomial maps. More precisely, following the construction of motivic Milnor fibers in Denef-Loeser [6] and [7], we introduce motivic reincarnations of global (Milnor) fibers of polynomial maps and give some methods for the calculations of their mixed Hodge numbers. Since by construction these mixed Hodge numbers contain the information on the monodromy at infinity of the map, we thereby determine its Jordan normal form. In particular, we will describe the numbers of Jordan blocks in the monodromy at infinity in terms of its Newton polyhedron at infinity.

In order to explain our results more precisely, we recall the definition and the basic properties of monodromies at infinity. After two fundamental papers [2] and [43], many authors studied the global behavior of polynomial maps $f\colon \mathbb{C}^n \longrightarrow \mathbb{C}$. For a polynomial map $f\colon \mathbb{C}^n \longrightarrow \mathbb{C}$, it is well-known that there exists a finite subset $B \subset \mathbb{C}$ such that the restriction

$$\mathbb{C}^n \setminus f^{-1}(B) \longrightarrow \mathbb{C} \setminus B \tag{1.1}$$

of f is a locally trivial fibration. We denote by B_f the smallest subset $B \subset \mathbb{C}$ satisfying this condition. Let $C_R = \{x \in \mathbb{C} \mid |x| = R\}$ $(R \gg 0)$ be a sufficiently large circle in $\mathbb{C}$ such that $B_f \subset \{x \in \mathbb{C} \mid |x| < R\}$. Then by restricting the locally trivial fibration $\mathbb{C}^n \setminus f^{-1}(B_f) \longrightarrow \mathbb{C} \setminus B_f$ to C_R we obtain a geometric monodromy automorphism $\Phi_f^{\infty}\colon f^{-1}(R) \xrightarrow{\sim} f^{-1}(R)$ and the linear maps

$$\Phi_j^{\infty}\colon H^j(f^{-1}(R);\mathbb{C}) \xrightarrow{\sim} H^j(f^{-1}(R);\mathbb{C}) \quad (j = 0, 1, \ldots) \tag{1.2}$$

associated to it, where the orientation of C_R is taken to be counter-clockwise as usual. We call Φ_j^{∞}'s the (cohomological) monodromies at infinity of f. Various formulas for their

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eigenvalues (i.e. the semisimple parts) were obtained by Libgober-Sperber [24] etc. Also, some important results on their nilpotent parts were obtained by García-López-Némethi [16] and Dimca-Saito [11] etc. For example, Dimca-Saito [11] obtained an upper bound of the sizes of Jordan blocks for the eigenvalue 1 in Φ_j^∞. For the special case $n = 2$, see [8] etc. However, to the best of our knowledge, the nilpotent parts have not yet been fully understood. The monodromies at infinity Φ_j^∞ are important, because after a basic result [32] of Neumann-Norbury, Dimca-Némethi [10] proved that the monodromy representations

$$\pi_1(\mathbb{C} \setminus B_f, c) \longrightarrow \mathrm{Aut}(H^j(f^{-1}(c); \mathbb{C})) \qquad (c \in \mathbb{C} \setminus B_f) \tag{1.3}$$

are completely determined by Φ_j^∞'s. In this paper, assuming that f is convenient and non-degenerate at infinity (see Definition 3.5) we describe the nilpotent parts (i.e. the Jordan normal forms) explicitly. Note that the second condition is satisfied by generic polynomials $f(x) \in \mathbb{C}[x_1, x_2, \dots, x_n]$. By the results of Broughton [2], f is tame at infinity (see Definition 3.1) and there exists a strong concentration $H^j(f^{-1}(R); \mathbb{C}) \simeq 0$ $(j \neq 0, n-1)$ of the cohomology groups of the generic fiber $f^{-1}(R)$ $(R \gg 0)$ of f. Since $\Phi_0^\infty = \mathrm{id}_{\mathbb{C}}$, Φ_{n-1}^∞ is the only non-trivial monodromy. Following [24], we call the convex hull of $\{0\}$ and the Newton polytope $NP(f)$ of f in $\mathbb{R}^n$ the Newton polyhedron of f at infinity and denote it by $\Gamma_\infty(f)$. Let $q_1, \dots, q_l$ (resp. $\gamma_1, \dots, \gamma_{l'}$) be the 0-dimensional (resp. 1-dimensional) faces of $\Gamma_\infty(f)$ such that $q_i \in \mathrm{Int}(\mathbb{R}_+^n)$ (resp. the relative interior $\mathrm{rel.int}(\gamma_i)$ of γ_i is contained in $\mathrm{Int}(\mathbb{R}_+^n)$). For each q_i (resp. γ_i), denote by $d_i > 0$ (resp. $e_i > 0$) its lattice distance $\mathrm{dist}(q_i, 0)$ (resp. $\mathrm{dist}(\gamma_i, 0)$) from the origin $0 \in \mathbb{R}^n$. For $1 \leq i \leq l'$, let Δ_i be the convex hull of $\{0\} \sqcup \gamma_i$ in $\mathbb{R}^n$. Then for $\lambda \in \mathbb{C} \setminus \{1\}$ and $1 \leq i \leq l'$ such that $\lambda^{e_i} = 1$ we set

$$n(\lambda)_i = \sharp\{v \in \mathbb{Z}^n \cap \mathrm{rel.int}(\Delta_i) \mid \mathrm{ht}(v, \gamma_i) = k\} + \sharp\{v \in \mathbb{Z}^n \cap \mathrm{rel.int}(\Delta_i) \mid \mathrm{ht}(v, \gamma_i) = e_i - k\}, \tag{1.4}$$

where k is the minimal positive integer satisfying $\lambda = \zeta_{e_i}^k$ $(\zeta_{e_i} = \exp(2\pi\sqrt{-1}/e_i))$ and for $v \in \mathbb{Z}^n \cap \mathrm{rel.int}(\Delta_i)$ we denote by $\mathrm{ht}(v, \gamma_i)$ the lattice height of v from the base γ_i of Δ_i. Then in Section 5 we prove the following result which describes the number of Jordan blocks for each fixed eigenvalue $\lambda \neq 1$ in Φ_{n-1}^∞. Recall that by the monodromy theorem the sizes of such Jordan blocks are bounded by n.

Theorem 1.1 *In the situation as above, for any $\lambda \in \mathbb{C}^* \setminus \{1\}$ we have*

(i) *The number of the Jordan blocks for the eigenvalue λ with the maximal possible size n in $\Phi_{n-1}^\infty \colon H^{n-1}(f^{-1}(R); \mathbb{C}) \xrightarrow{\sim} H^{n-1}(f^{-1}(R); \mathbb{C})$ $(R \gg 0)$ is equal to $\sharp\{q_i \mid \lambda^{d_i} = 1\}$.*

(ii) *The number of the Jordan blocks for the eigenvalue λ with size $n-1$ in Φ_{n-1}^∞ is equal to $\sum_{i \colon \lambda^{e_i}=1} n(\lambda)_i$.*

Namely the nilpotent parts for the eigenvalues $\lambda \neq 1$ in the monodromy at infinity Φ_{n-1}^∞ are determined by the lattice distances of the faces of $\Gamma_\infty(f)$ from the origin $0 \in \mathbb{R}^n$. The monodromy theorem asserts also that the sizes of the Jordan blocks for the eigenvalue 1 in Φ_{n-1}^∞ are bounded by $n-1$. In this case, we have the following result. Denote by Π_f the number of the lattice points on the 1-skeleton of $\partial\Gamma_\infty(f) \cap \mathrm{Int}(\mathbb{R}_+^n)$. We say also that $\gamma \prec \Gamma_\infty(f)$ is a face at infinity of $\Gamma_\infty(f)$ if $0 \notin \gamma$. For a face at infinity $\gamma \prec \Gamma_\infty(f)$, denote by $l^*(\gamma)$ the number of the lattice points on the relative interior $\mathrm{rel.int}(\gamma)$ of γ.

Theorem 1.2 (i) *The number of the Jordan blocks for the eigenvalue* 1 *with the maximal possible size* $n-1$ *in* Φ_{n-1}^{∞} *is* Π_f.

(ii) *The number of the Jordan blocks for the eigenvalue* 1 *with size* $n-2$ *in* Φ_{n-1}^{∞} *is equal to* $2\sum_{\gamma} l^*(\gamma)$, *where* γ *ranges through the faces of* $\Gamma_{\infty}(f)$ *at infinity such that* $\dim\gamma = 2$ *and* $\mathrm{rel.int}(\gamma) \subset \mathrm{Int}(\mathbb{R}_+^n)$. *In particular, this number is even.*

Roughly speaking, the nilpotent part for the eigenvalue 1 in the monodromy at infinity Φ_{n-1}^{∞} is determined by the convexity of the hypersurface $\partial\Gamma_{\infty}(f) \cap \mathrm{Int}(\mathbb{R}_+^n)$. Thus Theorem 1.1 and 1.2 generalize the well-known fact that the monodromies of quasi-homogeneous polynomials are semisimple. Moreover we will also give a general algorithm for computing the numbers of Jordan blocks with smaller sizes. See Section 5 for the detail.

This paper is organized as follows. In Section 2, after recalling some basic notions we give some generalizations of the results in Danilov-Khovanskii [5] which will be effectively used later. In Section 3, we recall some basic definitions on monodromies at infinity and review our new proof in [28] of Libgober-Sperber's theorem [24] on the semisimple parts of monodromies at infinity.

In Section 4, we prove global analogues of the results in Denef-Loeser [6] and [7]. Namely by mimicking their construction, we introduce motivic Milnor fibers at infinity and prove basic results. Note that after our introduction of motivic Milnor fibers at infinity in the preliminary version arXiv:0809.3149v7 of [28] Raibaut [35] introduced the same notion. Some deep results in Sabbah [36], [38] and [39] will be used to justify our arguments. Then in Section 5, by rewriting these results in terms of the Newton polyhedron at infinity $\Gamma_{\infty}(f)$ with the help of the results in Section 2, we prove some combinatorial formulas for the Jordan normal form of the monodromy at infinity Φ_{n-1}^{∞}. We obtain also a global analogue of the Steenbrink conjecture proved by Varchenko-Khovanskii [50] and Saito [41]. In Section 6, without assuming that f is tame at infinity, we prove some general results on the upper bounds for the sizes and the numbers of Jordan blocks in the monodromies at infinity Φ_j^{∞}. Finally in Section 7, we apply our methods also to local Milnor monodromies and obtain results completely parallel to Theorem 1.1 and 1.2 etc. We thus find a striking complete symmetry between local and global. Note that in the recent preprint [14] the results in this paper were generalized to the monodromies over complete intersection subvarieties in $\mathbb{C}^n$.

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2 Preliminary notions and results

In this section, we introduce basic notions and results which will be used in this paper. In this paper, we essentially follow the terminology of [9], [18] and [19]. For example, for a topological space X we denote by $\mathbf{D}^b(X)$ the derived category whose objects are bounded complexes of sheaves of $\mathbb{C}_X$-modules on X. For an algebraic variety X over $\mathbb{C}$, let $\mathbf{D}_c^b(X)$ be the full subcategory of $\mathbf{D}^b(X)$ consisting of constructible complexes of sheaves. In this case, for an abelian group G we denote by $\mathrm{CF}_G(X)$ the abelian group

of G-valued constructible functions on X. Let $\mathbb{C}(t)^* = \mathbb{C}(t) \setminus \{0\}$ be the multiplicative group of the function field $\mathbb{C}(t)$ of the scheme $\mathbb{C}$. In this paper, we consider $\mathrm{CF}_G(X)$ only for $G = \mathbb{Z}$ or $\mathbb{C}(t)^*$. For a G-valued constructible function $\rho\colon X \longrightarrow G$, by taking a stratification $X = \bigsqcup_\alpha X_\alpha$ of X such that $\rho|_{X_\alpha}$ is constant for any α, we set $\int_X \rho := \sum_\alpha \chi(X_\alpha) \cdot \rho(x_\alpha) \in G$, where x_α is a reference point in X_α. Then we can easily show that $\int_X \rho \in G$ does not depend on the choice of the stratification $X = \bigsqcup_\alpha X_\alpha$ of X. More generally, for any morphism $f\colon X \longrightarrow Y$ of algebraic varieties over $\mathbb{C}$ and $\rho \in \mathrm{CF}_G(X)$, we define the push-forward $\int_f \rho \in \mathrm{CF}_G(Y)$ of ρ by $(\int_f \rho)(y) := \int_{f^{-1}(y)} \rho$ for $y \in Y$. Now recall that for a non-constant regular function $f\colon X \longrightarrow \mathbb{C}$ on a variety X over $\mathbb{C}$ and the hypersurface $X_0 := \{x \in X \mid f(x) = 0\} \subset X$ there exists a nearby cycle functor

$$\psi_f\colon \mathbf{D}^b_c(X) \longrightarrow \mathbf{D}^b_c(X_0) \tag{2.1}$$

defined by Deligne (see [9, Section 4.2] for an excellent survey of this subject). As we see in the next proposition, the nearby cycle functor ψ_f generalizes the classical notion of Milnor fibers. In the above situation, for $x \in X_0$ denote by F_x the Milnor fiber of $f\colon X \longrightarrow \mathbb{C}$ at x (see for example [46] for a review on this subject).

Proposition 2.1 ([9, Proposition 4.2.2]) *For any $\mathcal{F} \in \mathbf{D}^b_c(X)$, $x \in X_0$ and $j \in \mathbb{Z}$, there exists a natural isomorphism*

$$H^j(F_x; \mathcal{F}) \simeq H^j(\psi_f(\mathcal{F}))_x. \tag{2.2}$$

By this proposition, we can study the cohomology groups $H^j(F_x; \mathbb{C})$ of the Milnor fiber F_x by using sheaf theory. Recall also that in the above situation we can define the Milnor monodromy operators

$$\Phi_{j,x}\colon H^j(F_x; \mathbb{C}) \xrightarrow{\sim} H^j(F_x; \mathbb{C}) \ (j = 0, 1, \ldots) \tag{2.3}$$

and the zeta function

$$\zeta_{f,x}(t) := \prod_{j=0}^{\infty} \det(\mathrm{id} - t\Phi_{j,x})^{(-1)^j} \in \mathbb{C}(t)^* \tag{2.4}$$

associated with it. This classical notion of Milnor monodromy zeta functions can be also generalized as follows.

Definition 2.2 Let $f\colon X \longrightarrow \mathbb{C}$ be a non-constant regular function on X and $X_0 := \{x \in X \mid f(x) = 0\}$ the hypersurface defined by it. Then for $\mathcal{F} \in \mathbf{D}^b_c(X)$ there exists a monodromy automorphism

$$\Phi(\mathcal{F})\colon \psi_f(\mathcal{F}) \xrightarrow{\sim} \psi_f(\mathcal{F}) \tag{2.5}$$

of $\psi_f(\mathcal{F})$ in $\mathbf{D}^b_c(X_0)$ (see e.g. [9, Section 4.2]). We define a $\mathbb{C}(t)^*$-valued constructible function $\zeta_f(\mathcal{F}) \in \mathrm{CF}_{\mathbb{C}(t)^*}(X_0)$ on X_0 by

$$\zeta_{f,x}(\mathcal{F})(t) := \prod_{j \in \mathbb{Z}} \det\left(\mathrm{id} - t\Phi(\mathcal{F})_{j,x}\right)^{(-1)^j} \in \mathbb{C}(t)^* \tag{2.6}$$

for $x \in X_0$, where $\Phi(\mathcal{F})_{j,x}\colon (H^j(\psi_f(\mathcal{F})))_x \xrightarrow{\sim} (H^j(\psi_f(\mathcal{F})))_x$ are induced by $\Phi(\mathcal{F})$.

For the proof of the following proposition, see for example, [9, p.170-173].

Proposition 2.3 *Let $\pi\colon Y \longrightarrow X$ be a proper morphism of algebraic varieties over $\mathbb{C}$ and $f\colon X \longrightarrow \mathbb{C}$ a non-constant regular function on X. Set $g := f \circ \pi\colon Y \longrightarrow \mathbb{C}$, $X_0 := \{x \in X \mid f(x) = 0\}$ and $Y_0 := \{y \in Y \mid g(y) = 0\} = \pi^{-1}(X_0)$. Then for any $\mathcal{G} \in \mathbf{D}_c^b(Y)$ we have*

$$\int_{\pi|_{Y_0}} \zeta_g(\mathcal{G}) = \zeta_f(R\pi_*\mathcal{G}) \tag{2.7}$$

in $\mathrm{CF}_{\mathbb{C}(t)^*}(X_0)$, where $\int_{\pi|_{Y_0}}\colon \mathrm{CF}_{\mathbb{C}(t)^*}(Y_0) \longrightarrow \mathrm{CF}_{\mathbb{C}(t)^*}(X_0)$ is the push-forward of $\mathbb{C}(t)^*$-valued constructible functions by $\pi|_{Y_0}\colon Y_0 \longrightarrow X_0$.

From now on, let us introduce our slight generalizations of the results in Danilov-Khovanskii [5].

Definition 2.4 Let $g(x) = \sum_{v\in\mathbb{Z}^n} a_v x^v$ ($a_v \in \mathbb{C}$) be a Laurent polynomial on $(\mathbb{C}^*)^n$.

(i) We call the convex hull of $\mathrm{supp}(g) := \{v \in \mathbb{Z}^n \mid a_v \neq 0\} \subset \mathbb{Z}^n$ in $\mathbb{R}^n$ the Newton polytope of g and denote it by $NP(g)$.

(ii) For $u \in (\mathbb{R}^n)^*$, we set $\Gamma(g; u) := \left\{v \in NP(g) \mid \langle u, v\rangle = \min_{w\in NP(g)}\langle u, w\rangle \right\}$.

(iii) For $u \in (\mathbb{R}^n)^*$, we define the u-part of g by $g^u(x) := \sum_{v\in\Gamma(g;u)} a_v x^v$.

Definition 2.5 ([21]) Let g be a Laurent polynomial on $(\mathbb{C}^*)^n$. Then we say that the hypersurface $Z^* = \{x \in (\mathbb{C}^*)^n \mid g(x) = 0\}$ of $(\mathbb{C}^*)^n$ is non-degenerate if for any $u \in (\mathbb{R}^n)^*$ the hypersurface $\{x \in (\mathbb{C}^*)^n \mid g^u(x) = 0\}$ is smooth and reduced.

In the sequel, let us fix an element $\tau = (\tau_1, \dots, \tau_n) \in T := (\mathbb{C}^*)^n$ and let g be a Laurent polynomial on $(\mathbb{C}^*)^n$ such that $Z^* = \{x \in (\mathbb{C}^*)^n \mid g(x) = 0\}$ is non-degenerate and invariant by the automorphism $l_\tau\colon (\mathbb{C}^*)^n \underset{\tau\times}{\overset{\sim}{\longrightarrow}} (\mathbb{C}^*)^n$ induced by the multiplication by τ. Set $\Delta = NP(g)$ and for simplicity assume that $\dim\Delta = n$. Then there exists $\beta \in \mathbb{C}$ such that $l_\tau^* g = g \circ l_\tau = \beta g$. This implies that for any vertex v of $\Delta = NP(g)$ we have $\tau^v = \tau_1^{v_1} \cdots \tau_n^{v_n} = \beta$. Moreover by the condition $\dim\Delta = n$ we see that $\tau_1, \tau_2, \dots, \tau_n$ are roots of unity. For $p, q \geq 0$ and $k \geq 0$, let $h^{p,q}(H_c^k(Z^*;\mathbb{C}))$ be the mixed Hodge number of $H_c^k(Z^*;\mathbb{C})$ and set

$$e^{p,q}(Z^*) = \sum_k (-1)^k h^{p,q}(H_c^k(Z^*;\mathbb{C})) \tag{2.8}$$

as in [5]. The above automorphism of $(\mathbb{C}^*)^n$ induces a morphism of mixed Hodge structures $l_\tau^*\colon H_c^k(Z^*;\mathbb{C}) \overset{\sim}{\longrightarrow} H_c^k(Z^*;\mathbb{C})$ and hence $\mathbb{C}$-linear automorphisms of the (p,q)-parts $H_c^k(Z^*;\mathbb{C})^{p,q}$ of $H_c^k(Z^*;\mathbb{C})$. For $\alpha \in \mathbb{C}$, let $h^{p,q}(H_c^k(Z^*;\mathbb{C}))_\alpha$ be the dimension of the α-eigenspace $H_c^k(Z^*;\mathbb{C})_\alpha^{p,q}$ of this automorphism of $H_c^k(Z^*;\mathbb{C})^{p,q}$ and set

$$e^{p,q}(Z^*)_\alpha = \sum_k (-1)^k h^{p,q}(H_c^k(Z^*;\mathbb{C}))_\alpha. \tag{2.9}$$

Since we have $l_\tau^r = \mathrm{id}_{Z^*}$ for some $r \gg 0$, these numbers are zero unless α is a root of unity. Obviously we have

$$e^{p,q}(Z^*) = \sum_{\alpha\in\mathbb{C}} e^{p,q}(Z^*)_\alpha, \qquad e^{p,q}(Z^*)_\alpha = e^{q,p}(Z^*)_{\overline{\alpha}}. \tag{2.10}$$

In this setting, along the lines of Danilov-Khovanskii [5] we can give an algorithm for computing these numbers $e^{p,q}(Z^*)_\alpha$ as follows. First of all, as in [5, Section 3] we can easily obtain the following result.

Proposition 2.6 *For $p, q \geq 0$ such that $p+q > n-1$, we have*

$$e^{p,q}(Z^*)_\alpha = \begin{cases} (-1)^{n+p+1}\binom{n}{p+1} & (\alpha = 1 \ \textit{and}\ p = q), \\ 0 & (\textit{otherwise}), \end{cases} \tag{2.11}$$

(we used the convention $\binom{a}{b} = 0$ $(0 \leq a < b)$ for binomial coefficients).

Proof. If $p+q > n-1$, we have $H_c^k(Z^*;\mathbb{C})^{p,q} = 0$ for $k \leq n-1$. Moreover for $p, q \geq 0$ such that $p+q > n-1$ and $k > n-1$ the Gysin homomorphism

$$\Theta_{p,q} \colon H_c^k(Z^*;\mathbb{C})^{p,q} \longrightarrow H_c^{k+2}((\mathbb{C}^*)^n;\mathbb{C})^{p+1,q+1} \tag{2.12}$$

is an isomorphism by [5, Proposition 3.2]. Since for such p, q and k there exists a commutative diagram

$$\begin{array}{ccc} H_c^k(Z^*;\mathbb{C})^{p,q} & \xrightarrow[\Theta_{p,q}]{\sim} & H_c^{k+2}((\mathbb{C}^*)^n;\mathbb{C})^{p+1,q+1} \\ \wr\big\downarrow l_\tau^* & & \wr\big\downarrow l_\tau^* \\ H_c^k(Z^*;\mathbb{C})^{p,q} & \xrightarrow[\Theta_{p,q}]{\sim} & H_c^{k+2}((\mathbb{C}^*)^n;\mathbb{C})^{p+1,q+1} \end{array} \tag{2.13}$$

and $l_\tau \colon (\mathbb{C}^*)^n \xrightarrow{\sim} (\mathbb{C}^*)^n$ is homotopic to the identity of $(\mathbb{C}^*)^n$, we obtain isomorphisms

$$H_c^k(Z^*;\mathbb{C})^{p,q}_\alpha \simeq H_c^{k+2}((\mathbb{C}^*)^n;\mathbb{C})^{p+1,q+1}_\alpha = \begin{cases} \mathbb{C}^{\binom{n}{p+1}} & (k = n+p-1,\ \alpha = 1 \text{ and } p = q), \\ 0 & (\text{otherwise}). \end{cases} \tag{2.14}$$

Then the result follows from the definition of $e^{p,q}(Z^*)_\alpha$. This completes the proof. □

For a vertex w of Δ, consider the translated polytope $\Delta^w := \Delta - w$ such that $0 \prec \Delta^w$ and $\tau^v = 1$ for any vertex v of Δ^w. Then for $\alpha \in \mathbb{C}$ and $k \geq 0$ set

$$l^*(k\Delta)_\alpha = \sharp\{v \in \mathrm{Int}(k\Delta^w) \cap \mathbb{Z}^n \mid \tau^v = \alpha\} \in \mathbb{Z}_+ := \mathbb{Z}_{\geq 0} \tag{2.15}$$

and

$$l(k\Delta)_\alpha = \sharp\{v \in (k\Delta^w) \cap \mathbb{Z}^n \mid \tau^v = \alpha\} \in \mathbb{Z}_+. \tag{2.16}$$

We can easily see that these numbers $l^*(k\Delta)_\alpha$ and $l(k\Delta)_\alpha$ do not depend on the choice of the vertex w of Δ. Next, define two formal power series $P_\alpha(\Delta;t) = \sum_{i\geq 0}\varphi_{\alpha,i}(\Delta)t^i$ and $Q_\alpha(\Delta;t) = \sum_{i\geq 0}\psi_{\alpha,i}(\Delta)t^i$ by

$$P_\alpha(\Delta;t) = (1-t)^{n+1}\left\{\sum_{k\geq 0} l^*(k\Delta)_\alpha t^k\right\} \tag{2.17}$$

and

$$Q_\alpha(\Delta;t) = (1-t)^{n+1}\left\{\sum_{k\geq 0} l(k\Delta)_\alpha t^k\right\} \tag{2.18}$$

respectively. Then we can easily show that $P_\alpha(\Delta; t)$ is actually a polynomial as in [5, Section 4.4]. Moreover as in Macdonald [26], we can easily prove that for any $\alpha \in \mathbb{C}^*$ the function $h_{\Delta,\alpha}(k) := l(k\Delta)_{\alpha^{-1}}$ of $k \geq 0$ is a polynomial of degree n with coefficients in $\mathbb{Q}$. By a straightforward generalization of the Ehrhart reciprocity proved by [26], we obtain also an equality

$$h_{\Delta,\alpha}(-k) = (-1)^n l^*(k\Delta)_\alpha \tag{2.19}$$

for $k > 0$. By an elementary computation (see [5, Remark 4.6]), this implies that we have

$$\varphi_{\alpha,i}(\Delta) = \psi_{\alpha^{-1},n+1-i}(\Delta) \qquad (i \in \mathbb{Z}). \tag{2.20}$$

In particular, $Q_\alpha(\Delta; t) = \sum_{i\geq 0} \psi_{\alpha,i}(\Delta) t^i$ is a polynomial for any $\alpha \in \mathbb{C}^*$.

Theorem 2.7 *In the situation as above, we have*

$$\sum_q e^{p,q}(Z^*)_\alpha = \begin{cases} (-1)^{p+n+1}\binom{n}{p+1} + (-1)^{n+1}\varphi_{\alpha,n-p}(\Delta) & (\alpha = 1), \\ (-1)^{n+1}\varphi_{\alpha,n-p}(\Delta) & (\alpha \neq 1). \end{cases} \tag{2.21}$$

Proof. Let Σ_1 be the dual fan of Δ in $\mathbb{R}^n$. Then we can construct a subdivision Σ of Σ_1 such that the toric variety X_Σ associated with it is smooth and projective. Moreover, there exists a T-equivariant line bundle $\mathcal{O}_{X_\Sigma}(\Delta)$ on X_Σ whose global section $\varGamma(X_\Sigma; \mathcal{O}_{X_\Sigma}(\Delta))$ is naturally isomorphic to the space $\{\sum_{v\in\Delta\cap\mathbb{Z}^n} a_v x^v \mid a_v \in \mathbb{C}\}$ of Laurent polynomials with support in $\Delta \cap \mathbb{Z}^n$ (see [5, Section 2] and [33, Section 2.1] etc.). Since the Laurent polynomial g is a section of $\mathcal{O}_{X_\Sigma}(\Delta)$, we obtain an isomorphism $\mathcal{O}_{X_\Sigma}(\Delta) \simeq \mathcal{O}_{X_\Sigma}(\overline{Z^*})$. Then by using the isomorphism

$$\varGamma(X_\Sigma; \mathcal{O}_{X_\Sigma}(\Delta)) \underset{\mathbf{A}}{\xrightarrow{\sim}} \varGamma(X_\Sigma; \mathcal{O}_{X_\Sigma}(\overline{Z^*})) \tag{2.22}$$

and the pull-back of the meromorphic functions in $\varGamma(X_\Sigma; \mathcal{O}_{X_\Sigma}(\overline{Z^*}))$ by l_τ, we can define an action l_τ^* of $\tau \in (\mathbb{C}^*)^n$ on $\varGamma(X_\Sigma; \mathcal{O}_{X_\Sigma}(\Delta)) \simeq \{\sum_{v\in\Delta\cap\mathbb{Z}^n} a_v x^v \mid a_v \in \mathbb{C}\}$. Note that this action l_τ^* is different from the one constructed in [33, Section 2.1 and 2.2] by using the T-equivariance of the line bundle $\mathcal{O}_{X_\Sigma}(\Delta)$. From now on, we shall describe the action l_τ^* explicitly. For an n-dimensional cone $\sigma \in \Sigma$, let $v_\sigma \prec \Delta$ be the 0-dimensional supporting face of σ in Δ and $U_\sigma \simeq \mathbb{C}^n_y$ the affine open subset of X_Σ which corresponds to σ. More precisely we set $U_\sigma = \operatorname{Spec}(\mathbb{C}[\sigma^\vee \cap \mathbb{Z}^n])$. Then on $U_\sigma \simeq \mathbb{C}^n_y$ we have

$$g(y) = y_1^{a_1} \cdots y_n^{a_n} \times g_\sigma(y) \qquad (a_i \in \mathbb{Z}), \tag{2.23}$$

where g_σ is a polynomial such that $NP(g_\sigma) = \Delta^{v_\sigma} = \Delta - v_\sigma$. Namely, in U_σ the hypersurface $\overline{Z^*} \subset X_\Sigma$ is defined by $\overline{Z^*} = \{g_\sigma = 0\}$. Hence there exists an isomorphism

$$\varGamma(U_\sigma; \mathcal{O}_{X_\Sigma}) \underset{\mathbf{B}}{\xrightarrow{\sim}} \varGamma(U_\sigma; \mathcal{O}_{X_\Sigma}(\overline{Z^*})) \tag{2.24}$$

given by

$$\sum_{v\in\sigma^\vee\cap\mathbb{Z}^n} a_v x^v \longmapsto \frac{1}{g_\sigma} \sum_{v\in\sigma^\vee\cap\mathbb{Z}^n} a_v x^v. \tag{2.25}$$

Since we have $l_\tau^* g_\sigma = g_\sigma \circ l_\tau = g_\sigma$ by the construction of g_σ, via the isomorphism $\mathbf{B}$, the action l_τ^* of $\tau \in (\mathbb{C}^*)^n$ on $\varGamma(U_\sigma; \mathcal{O}_{X_\Sigma}(\overline{Z^*}))$ corresponds to the automorphism of $\varGamma(U_\sigma; \mathcal{O}_{X_\Sigma}) \simeq \mathbb{C}[\sigma^\vee \cap \mathbb{Z}^n]$ defined by

$$\sum_{v\in\sigma^\vee\cap\mathbb{Z}^n} a_v x^v \longmapsto \sum_{v\in\sigma^\vee\cap\mathbb{Z}^n} a_v \tau^v x^v. \tag{2.26}$$

On the other hand, there exists also a natural injection

$$\mathit{\Gamma}(X_\Sigma; \mathcal{O}_{X_\Sigma}(\Delta)) \underset{\mathbf{C}}{\hookrightarrow} \mathit{\Gamma}(U_\sigma; \mathcal{O}_{X_\Sigma}) \tag{2.27}$$

given by

$$\sum_{v \in \Delta \cap \mathbb{Z}^n} a_v x^v \longmapsto \sum_{v \in \sigma^\vee \cap \mathbb{Z}^n} a_v x^{v - v_\sigma}. \tag{2.28}$$

Then by the commutative diagram

$$\begin{array}{ccc} \mathit{\Gamma}(X_\Sigma; \mathcal{O}_{X_\Sigma}(\Delta)) & \xhookrightarrow{\quad\mathbf{C}\quad} & \mathit{\Gamma}(U_\sigma; \mathcal{O}_{X_\Sigma}) \\ \wr\big\downarrow \mathbf{A} & & \wr\big\downarrow \mathbf{B} \\ \mathit{\Gamma}(X_\Sigma; \mathcal{O}_{X_\Sigma}(\overline{Z^*})) & \xhookrightarrow{\qquad\quad} & \mathit{\Gamma}(U_\sigma; \mathcal{O}_{X_\Sigma}(\overline{Z^*})), \end{array} \tag{2.29}$$

we see that the action l_τ^* on $\mathit{\Gamma}(X_\Sigma; \mathcal{O}_{X_\Sigma}(\Delta))$ is given by

$$\sum_{v \in \Delta \cap \mathbb{Z}^n} a_v x^v \longmapsto \sum_{v \in \Delta \cap \mathbb{Z}^n} a_v \tau^{v - v_\sigma} x^v. \tag{2.30}$$

Note that this morphism does not depend on the choice of the n-dimensional cone $\sigma \in \Sigma$. From now on, we will describe also a natural action of $\tau \in T = (\mathbb{C}^*)^n$ on $\mathit{\Gamma}(X_\Sigma; \mathcal{O}_{X_\Sigma}(k\Delta))$ for $k \geq 1$. For $k \geq 1$, let g_k be a Laurent polynomial on $(\mathbb{C}^*)^n$ such that $NP(g_k) = k\Delta$ and $Z_k^* = \{x \in (\mathbb{C}^*)^n \mid g_k(x) = 0\}$ is non-degenerate and stable by the automorphism $l_\tau \colon (\mathbb{C}^*)^n \xrightarrow{\sim} (\mathbb{C}^*)^n$. Such a Laurent polynomial g_k always exists (see Lemma 5.2 below). Since we have $\mathcal{O}_{X_\Sigma}(\Delta)^{\otimes k} \simeq \mathcal{O}_{X_\Sigma}(k\Delta)$, the k-th power g^k of the Laurent polynomial g is a section of $\mathcal{O}_{X_\Sigma}(k\Delta)$ satisfying the condition $\mathrm{div} g^k = k\overline{Z^*}$. Therefore we obtain isomorphisms

$$\mathcal{O}_{X_\Sigma}(k\Delta) \simeq \mathcal{O}_{X_\Sigma}(\overline{Z_k^*}) \simeq \mathcal{O}_{X_\Sigma}(k\overline{Z^*}) \tag{2.31}$$

and the Weil divisors $\overline{Z_k^*}$ and $k\overline{Z^*}$ are naturally equivalent. Now let $\sigma \in \Sigma$ be an n-dimensional cone. Then on $U_\sigma \simeq \mathbb{C}_y^n$ we have

$$g^k(y) = y_1^{ka_1} \cdots y_n^{ka_n} \times g_\sigma^k(y), \tag{2.32}$$

$$g_k(y) = y_1^{ka_1} \cdots y_n^{ka_n} \times (g_k)_\sigma(y), \tag{2.33}$$

where $(g_k)_\sigma$ is an l_τ^*-invariant polynomial on $U_\sigma \simeq \mathbb{C}_y^n$. From this, we see that the rational function $\frac{g^k}{g_k}$ on X_Σ is l_τ^*-invariant and

$$\mathrm{div}\left(\frac{g^k}{g_k}\right) = k\overline{Z^*} - \overline{Z_k^*} \tag{2.34}$$

on the whole X_Σ. Then there exists an isomorphism

$$\begin{array}{ccc} \mathit{\Gamma}(X_\Sigma; \mathcal{O}_{X_\Sigma}(k\overline{Z^*})) & \underset{\mathbf{D}}{\xrightarrow{\sim}} & \mathit{\Gamma}(X_\Sigma; \mathcal{O}_{X_\Sigma}(\overline{Z_k^*})) \\ \uplus & & \uplus \\ \varphi & \longmapsto & \dfrac{g^k}{g_k} \times \varphi \end{array} \tag{2.35}$$

and a commutative diagram

$$\begin{tikzcd}
\mathit{\Gamma}(X_\Sigma; \mathcal{O}_{X_\Sigma}(k\overline{Z^*})) \arrow[r, "\sim", "l_\tau^*"'] \arrow[d, "\mathbf{D}"', "\wr"] & \mathit{\Gamma}(X_\Sigma; \mathcal{O}_{X_\Sigma}(k\overline{Z^*})) \arrow[d, "\mathbf{D}"', "\wr"] \\
\mathit{\Gamma}(X_\Sigma; \mathcal{O}_{X_\Sigma}(\overline{Z_k^*})) \arrow[r, "\sim", "l_\tau^*"'] \arrow[d, "\wr"] & \mathit{\Gamma}(X_\Sigma; \mathcal{O}_{X_\Sigma}(\overline{Z_k^*})) \arrow[d, "\wr"] \\
\mathit{\Gamma}(X_\Sigma; \mathcal{O}_{X_\Sigma}(k\Delta)) \arrow[r, "\sim"] & \mathit{\Gamma}(X_\Sigma; \mathcal{O}_{X_\Sigma}(k\Delta)),
\end{tikzcd} \tag{2.36}$$

where the upper and middle horizontal arrows are the pull-backs of meromorphic functions on X_Σ by l_τ and by taking a vertex w of $k\Delta$ the bottom horizontal arrow is defined by

$$\sum_{v \in k\Delta \cap \mathbb{Z}^n} a_v x^v \longmapsto \sum_{v \in k\Delta \cap \mathbb{Z}^n} a_v \tau^{v-w} x^v. \tag{2.37}$$

Moreover, let $D_1, \dots, D_m$ be the (smooth) toric divisors on X_Σ such that $X_\Sigma \setminus (\bigcup_{i=1}^m D_i) = (\mathbb{C}^*)^n$ and for $0 \leq p \leq n$ set $D = \bigcup_{i=1}^m D_i$ and

$$\Omega^p_{(X_\Sigma, D)} = \mathrm{Ker}\left[\Omega^p_{X_\Sigma} \longrightarrow \bigoplus_{i=1}^m \Omega^p_{D_i} \right] \tag{2.38}$$

as in [5, Section 1.11]. Then there exists a well-known isomorphism

$$\bigwedge^p \mathbb{Z}^n \otimes_{\mathbb{Z}} \mathcal{O}_{X_\Sigma}(-\sum_{i=1}^m D_i) \overset{\sim}{\longrightarrow} \Omega^p_{(X_\Sigma, D)} \tag{2.39}$$

given by

$$(v_1 \wedge \cdots \wedge v_p) \otimes \varphi \longmapsto \varphi \times \frac{dx^{v_1}}{x^{v_1}} \wedge \cdots \wedge \frac{dx^{v_p}}{x^{v_p}}, \tag{2.40}$$

where $v_i \in \mathbb{Z}^n$ and $\varphi \in \mathcal{O}_{X_\Sigma}(-\sum_{i=1}^m D_i) \subset \mathcal{O}_{X_\Sigma}$. Since we have

$$l_\tau^* \left(\frac{dx^{v_1}}{x^{v_1}} \wedge \cdots \wedge \frac{dx^{v_p}}{x^{v_p}} \right) = \frac{dx^{v_1}}{x^{v_1}} \wedge \cdots \wedge \frac{dx^{v_p}}{x^{v_p}}, \tag{2.41}$$

for any $k \geq 1$ and $0 \leq p \leq n$ we obtain a commutative diagram

$$\begin{tikzcd}
\mathit{\Gamma}(X_\Sigma; \Omega^p_{(X_\Sigma,D)}(k\overline{Z^*})) \arrow[r, "\sim", "l_\tau^*"'] \arrow[d, "\wr"] & \mathit{\Gamma}(X_\Sigma; \Omega^p_{(X_\Sigma,D)}(k\overline{Z^*})) \arrow[d, "\wr"] \\
\bigwedge^p \mathbb{Z}^n \otimes_{\mathbb{Z}} \left\{ \sum_{v \in \mathrm{Int}(k\Delta) \cap \mathbb{Z}^n} a_v x^v \right\} \arrow[r, "\sim"] & \bigwedge^p \mathbb{Z}^n \otimes_{\mathbb{Z}} \left\{ \sum_{v \in \mathrm{Int}(k\Delta) \cap \mathbb{Z}^n} a_v x^v \right\},
\end{tikzcd} \tag{2.42}$$

where we set $\Omega^p_{(X_\Sigma,D)}(k\overline{Z^*}) = \Omega^p_{(X_\Sigma,D)} \otimes_{\mathcal{O}_{X_\Sigma}} \mathcal{O}_{X_\Sigma}(k\overline{Z^*})$ and by taking a vertex w of $k\Delta$ the bottom horizontal arrow is induced by

$$\sum_{v \in \mathrm{Int}(k\Delta) \cap \mathbb{Z}^n} a_v x^v \longmapsto \sum_{v \in \mathrm{Int}(k\Delta) \cap \mathbb{Z}^n} a_v \tau^{v-w} x^v. \tag{2.43}$$

By using this explicit description of

$$l_\tau^* \colon \mathit{\Gamma}(X_\Sigma; \Omega^p_{(X_\Sigma,D)}(k\overline{Z^*})) \overset{\sim}{\longrightarrow} \mathit{\Gamma}(X_\Sigma; \Omega^p_{(X_\Sigma,D)}(k\overline{Z^*})), \tag{2.44}$$

the assertion can be proved just by following the proof for the formula in [5, Section 4.4]. This completes the proof. □

With Proposition 2.6 and Theorem 2.7 at hands, we can now easily calculate the numbers $e^{p,q}(Z^*)_\alpha$ on the non-degenerate hypersurface $Z^* \subset (\mathbb{C}^*)^n$ for any $\alpha \in \mathbb{C}$ as in [5, Section 5.2]. Indeed for a projective toric compactification X of $(\mathbb{C}^*)^n$ such that the closure $\overline{Z^*}$ of Z^* in X is smooth, the variety $\overline{Z^*}$ is smooth projective and hence there exists a perfect pairing

$$H^{p,q}(\overline{Z^*};\mathbb{C})_\alpha \times H^{n-1-p,n-1-q}(\overline{Z^*};\mathbb{C})_{\alpha^{-1}} \longrightarrow \mathbb{C} \tag{2.45}$$

for any $p, q \geq 0$ and $\alpha \in \mathbb{C}^*$ (see for example [51, Section 5.3.2]). Therefore, we obtain equalities $e^{p,q}(\overline{Z^*})_\alpha = e^{n-1-p,n-1-q}(\overline{Z^*})_{\alpha^{-1}}$ which are necessary to proceed the algorithm in [5, Section 5.2]. We have also the following analogue of [5, Proposition 5.8].

Proposition 2.8 *For any $\alpha \in \mathbb{C}$ and $p > 0$ we have*

$$e^{p,0}(Z^*)_\alpha = e^{0,p}(Z^*)_{\overline{\alpha}} = (-1)^{n-1} \sum_{\substack{\Gamma \prec \Delta \\ \dim\Gamma = p+1}} l^*(\Gamma)_\alpha. \tag{2.46}$$

The following result is an analogue of [5, Corollary 5.10]. For $\alpha \in \mathbb{C}$, denote by $\Pi(\Delta)_\alpha$ the number of the lattice points $v = (v_1, \dots, v_n)$ on the 1-skeleton of $\Delta^w = \Delta - w$ such that $\tau^v = \alpha$, where w is a vertex of Δ.

Proposition 2.9 *In the situation as above, for any $\alpha \in \mathbb{C}^*$ we have*

$$e^{0,0}(Z^*)_\alpha = \begin{cases} (-1)^{n-1}\left(\Pi(\Delta)_1 - 1\right) & (\alpha = 1), \\ (-1)^{n-1}\Pi(\Delta)_{\alpha^{-1}} & (\alpha \neq 1). \end{cases} \tag{2.47}$$

Proof. By Theorem 2.7, Proposition 2.8 and the equality (2.20), the assertion can be proved as in the proof [5, Corollary 5.10]. □

For a vertex w of Δ, we define a closed convex cone $\mathrm{Con}(\Delta, w)$ by $\mathrm{Con}(\Delta, w) = \{r \cdot (v - w) \mid r \in \mathbb{R}_+,\ v \in \Delta\} \subset \mathbb{R}^n$.

Definition 2.10 Let Δ be an n-dimensional integral polytope in $(\mathbb{R}^n, \mathbb{Z}^n)$.

(i) (see [5, Section2.3]) We say that Δ is prime if for any vertex w of Δ the cone $\mathrm{Con}(\Delta, w)$ is generated by a basis of $\mathbb{R}^n$.

(ii) We say that Δ is pseudo-prime if for any 1-dimensional face $\gamma \prec \Delta$ the number of the 2-dimensional faces $\gamma' \prec \Delta$ such that $\gamma \prec \gamma'$ is $n - 1$.

By definition, prime polytopes are pseudo-prime. Moreover any face of a pseudo-prime polytope is again pseudo-prime.

Definition 2.11 ([5]) Let Δ and Δ' be two n-dimensional integral polytopes in $(\mathbb{R}^n, \mathbb{Z}^n)$. We denote by $\mathrm{som}(\Delta)$ (resp. $\mathrm{som}(\Delta')$) the set of vertices of Δ (resp. Δ'). Then we say that Δ' majorizes Δ if there exists a map $\Psi\colon \mathrm{som}(\Delta') \longrightarrow \mathrm{som}(\Delta)$ such that $\mathrm{Con}(\Delta, \Psi(w)) \subset \mathrm{Con}(\Delta', w)$ for any vertex w of Δ'.

For an integral polytope Δ in $(\mathbb{R}^n, \mathbb{Z}^n)$, we denote by X_Δ the toric variety associated with the dual fan of Δ. Recall that if Δ' majorizes Δ there exists a natural morphism $X_{\Delta'} \longrightarrow X_\Delta$.

Proposition 2.12 *Let Δ and $Z^*_\Delta = Z^*$ with an action of l_τ be as above. Assume that an n-dimensional integral polytope Δ' in $(\mathbb{R}^n, \mathbb{Z}^n)$ majorizes Δ by the map $\Psi\colon \mathrm{som}(\Delta') \longrightarrow \mathrm{som}(\Delta)$. Then for the closure $\overline{Z^*}$ of Z^* in $X_{\Delta'}$ we have*

$$\begin{aligned}\sum_q e^{p,q}(\overline{Z^*})_1 &= \sum_{\Gamma \prec \Delta'} (-1)^{\dim\Gamma + p+1} \left\{ \binom{\dim\Gamma}{p+1} - \binom{b_\Gamma}{p+1} \right\} \\ &\quad + \sum_{\Gamma \prec \Delta'} (-1)^{\dim\Gamma+1} \sum_{i=0}^{\min\{b_\Gamma, p\}} \binom{b_\Gamma}{i} (-1)^i \varphi_{1, \dim\Psi(\Gamma)-p+i}(\Psi(\Gamma)), \quad (2.48)\end{aligned}$$

where for $\Gamma \prec \Delta'$ we set $b_\Gamma = \dim\Gamma - \dim\Psi(\Gamma)$.

Proof. Since the toric variety $X_{\Delta'}$ has the decomposition $X_{\Delta'} = \bigsqcup_{\Gamma \prec \Delta'} T_\Gamma$ into T-orbits $T_\Gamma \simeq (\mathbb{C}^*)^{\dim\Gamma}$ and $\overline{Z^*} \cap T_\Gamma \simeq (\mathbb{C}^*)^{b_\Gamma} \times Z^*_{\Psi(\Gamma)}$, by Theorem 2.7 we have

$$\sum_q e^{p,q}(\overline{Z^*})_1 = \sum_q \sum_{\Gamma \prec \Delta'} e^{p,q}((\mathbb{C}^*)^{b_\Gamma} \times Z^*_{\Psi(\Gamma)})_1 \quad (2.49)$$

$$= \sum_{\Gamma \prec \Delta'} \sum_{i=0}^{\min\{b_\Gamma, p\}} \binom{b_\Gamma}{i} (-1)^{i+b_\Gamma} \sum_q e^{p-i,q-i}(Z^*_{\Psi(\Gamma)})_1 \quad (2.50)$$

$$\begin{aligned}&= \sum_{\Gamma \prec \Delta'} \sum_{i=0}^{\min\{b_\Gamma, p\}} \binom{b_\Gamma}{i} (-1)^{i+b_\Gamma} \times (-1)^{\dim\Psi(\Gamma)+1} \\ &\quad \times \left\{ (-1)^{p-i} \binom{\dim\Psi(\Gamma)}{p+1-i} + \varphi_{1,\dim\Psi(\Gamma)-p+i}(\Psi(\Gamma)) \right\}. \quad (2.51)\end{aligned}$$

Then the result follows from the simple calculations

$$\binom{\dim\Gamma}{p+1} = \binom{\dim\Psi(\Gamma) + b_\Gamma}{p+1} = \sum_{i=0}^{\min\{b_\Gamma, p\}} \binom{\dim\Psi(\Gamma)}{p+1-i} \binom{b_\Gamma}{i} + \binom{b_\Gamma}{p+1}. \quad (2.52)$$

□

From now on, we assume that $\Delta = NP(g)$ is pseudo-prime. Let Σ be the dual fan of Δ and X_Σ the toric variety associated to it. Then except finite points X_Σ is an orbifold and the closure $\overline{Z^*}$ of Z^* in X_Σ does not intersect such points by the non-degeneracy of g. Hence $\overline{Z^*}$ is an orbifold i.e. quasi-smooth in the sense of [5, Proposition 2.4]. In particular, there exists a Poincaré duality isomorphism

$$[H^{p,q}(\overline{Z^*}; \mathbb{C})_\alpha]^* \simeq H^{n-1-p, n-1-q}(\overline{Z^*}; \mathbb{C})_{\alpha^{-1}} \quad (2.53)$$

for any $\alpha \in \mathbb{C}^*$ (see for example [3] and [18, Corollary 8.2.22]). Then by slightly generalizing the arguments in [5] we obtain the following analogue of [5, Section 5.5 and Theorem 5.6].

Proposition 2.13 *In the situation as above, for any $\alpha \in \mathbb{C} \setminus \{1\}$ and $p, q \geq 0$, we have*

$$e^{p,q}(\overline{Z^*})_\alpha = \begin{cases} -\sum_{\Gamma \prec \Delta} (-1)^{\dim \Gamma} \varphi_{\alpha, \dim\Gamma - p}(\Gamma) & (p+q = n-1), \\ 0 & (\textit{otherwise}), \end{cases} \tag{2.54}$$

$$e^{p,q}(Z^*)_\alpha = (-1)^{n+p+q} \sum_{\substack{\Gamma \prec \Delta \\ \dim\Gamma = p+q+1}} \left\{ \sum_{\Gamma' \prec \Gamma} (-1)^{\dim \Gamma'} \varphi_{\alpha, \dim\Gamma' - p}(\Gamma') \right\}. \tag{2.55}$$

Proposition 2.14 *In the situation as above, we have*

(i) *For $p, q \geq 0$ such that $p \neq q$, we have*

$$e^{p,q}(\overline{Z^*})_1 = \begin{cases} -\sum_{\Gamma \prec \Delta} (-1)^{\dim \Gamma} \varphi_{1, \dim\Gamma - \max\{p,q\}}(\Gamma) & (p+q = n-1), \\ 0 & (\textit{otherwise}). \end{cases} \tag{2.56}$$

(ii) *For $p \geq 0$, we have*

$$e^{p,p}(\overline{Z^*})_1 = \begin{cases} (-1)^{p+1} \sum_{\substack{\Gamma \prec \Delta \\ \dim\Gamma \geq p+1}} (-1)^{\dim\Gamma} \binom{\dim\Gamma}{p+1} & (2p > n-1), \\ (-1)^{n-p} \sum_{\substack{\Gamma \prec \Delta \\ \dim\Gamma \geq n-p}} (-1)^{\dim\Gamma} \binom{\dim\Gamma}{n-p} & (2p < n-1), \\ \sum_{\Gamma \prec \Delta} (-1)^{\dim\Gamma} \left\{ (-1)^{p+1} \binom{\dim\Gamma}{p+1} - \varphi_{1, \dim\Gamma - p}(\Gamma) \right\} & (2p = n-1). \end{cases} \tag{2.57}$$

From this proposition and the proof of [5, Theorem 5.6], we obtain also the formula for $e^{p,q}(Z^*)_1$. For $\alpha \in \mathbb{C} \setminus \{1\}$ and a face $\Gamma \prec \Delta$, set $\widetilde{\varphi}_\alpha(\Gamma) = \sum_{i=0}^{\dim\Gamma} \varphi_{\alpha,i}(\Gamma)$. Then Proposition 2.13 can be rewritten as follows.

Corollary 2.15 *For any $\alpha \in \mathbb{C} \setminus \{1\}$ and $r \geq 0$, we have*

$$\sum_{p+q=r} e^{p,q}(Z^*)_\alpha = (-1)^{n+r} \sum_{\substack{\Gamma \prec \Delta \\ \dim\Gamma = r+1}} \left\{ \sum_{\Gamma' \prec \Gamma} (-1)^{\dim\Gamma'} \widetilde{\varphi}_\alpha(\Gamma') \right\}. \tag{2.58}$$

The following lemma will be used in Section 7.

Lemma 2.16 *Let γ be a d-dimensional prime polytope. Then for any $0 \leq p \leq d$ we have*

$$\sum_{\Gamma \prec \gamma} (-1)^{\dim\Gamma} \binom{\dim\Gamma}{p} = \sum_{\Gamma \prec \gamma} (-1)^{d + \dim\Gamma} \binom{\dim\Gamma}{d-p}. \tag{2.59}$$

Proof. For a polytope Δ, denote the number of the j-dimensional faces of Δ by $f_{\Delta,j}$ and set $f_{\Delta,-1} = 1$. Let $\gamma^\vee$ be the dual polytope of γ. Then $\gamma^\vee$ is simplicial and we have $f_{\gamma^\vee,j} = f_{\gamma,d-1-j}$ for any $0 \leq j \leq d$. Hence (2.59) follows from the Dehn-Sommerville equations for simplicial polytopes. □

3 Semisimple part of monodromies at infinity

In this section, we recall some basic definitions on monodromies at infinity and review our new proof in [28] of Libgober-Sperber's theorem [24]. Let $f(x)$ be a polynomial on $\mathbb{C}^n$. Then as we explained in Introduction, there exist a locally trivial fibration $\mathbb{C}^n \setminus f^{-1}(B_f) \longrightarrow \mathbb{C} \setminus B_f$ and the linear maps

$$\Phi_j^\infty \colon H^j(f^{-1}(R);\mathbb{C}) \xrightarrow{\sim} H^j(f^{-1}(R);\mathbb{C}) \quad (j = 0, 1, \dots) \tag{3.1}$$

($R \gg 0$) associated to it. To study the monodromies at infinity Φ_j^∞, we often impose the following natural condition.

Definition 3.1 ([21]) Let $\partial f \colon \mathbb{C}^n \longrightarrow \mathbb{C}^n$ be the map defined by $\partial f(x) = (\partial_1 f(x), \dots, \partial_n f(x))$. Then we say that f is tame at infinity if the restriction $(\partial f)^{-1}(B(0;\varepsilon)) \longrightarrow B(0;\varepsilon)$ of ∂f to a sufficiently small ball $B(0;\varepsilon)$ centered at the origin $0 \in \mathbb{C}^n$ is proper.

The following result is fundamental in the study of monodromies at infinity.

Theorem 3.2 (Broughton [2] and Siersma-Tibăr [43]) *Assume that f is tame at infinity. Then the generic fiber $f^{-1}(c)$ ($c \in \mathbb{C} \setminus B_f$) has the homotopy type of the bouquet of $(n-1)$-spheres. In particular, we have*

$$H^j(f^{-1}(c);\mathbb{C}) = 0 \quad (j \neq 0, n-1). \tag{3.2}$$

By this theorem if f is tame at infinity, then Φ_{n-1}^∞ is the only non-trivial monodromy at infinity and its characteristic polynomial is calculated by the following zeta function $\zeta_f^\infty(t) \in \mathbb{C}(t)^*$.

Definition 3.3 We define the monodromy zeta function at infinity $\zeta_f^\infty(t)$ of f by

$$\zeta_f^\infty(t) := \prod_{j=0}^{\infty} \det(\mathrm{id} - t\Phi_j^\infty)^{(-1)^j} \in \mathbb{C}(t)^*. \tag{3.3}$$

Definition 3.4 ([24]) We call the convex hull of $\{0\} \cup NP(f)$ in $\mathbb{R}^n$ the Newton polyhedron at infinity of f and denote it by $\Gamma_\infty(f)$.

For a subset $S \subset \{1, 2, \dots, n\}$, let us set

$$\mathbb{R}^S := \{v = (v_1, v_2, \dots, v_n) \in \mathbb{R}^n \mid v_i = 0 \text{ for } i \notin S\} \simeq \mathbb{R}^{\sharp S}. \tag{3.4}$$

We set also $\Gamma_\infty^S(f) = \Gamma_\infty(f) \cap \mathbb{R}^S$. Recall that f is convenient if we have $\dim \Gamma_\infty^S(f) = \sharp S$ for any $S \subset \{1, 2, \dots, n\}$.

Definition 3.5 ([21]) We say that $f(x) = \sum_{v \in \mathbb{Z}_+^n} a_v x^v$ ($a_v \in \mathbb{C}$) is non-degenerate at infinity if for any face γ of $\Gamma_\infty(f)$ such that $0 \notin \gamma$ the complex hypersurface $\{x \in (\mathbb{C}^*)^n \mid f_\gamma(x) = 0\}$ in $(\mathbb{C}^*)^n$ is smooth and reduced, where we set $f_\gamma(x) = \sum_{v \in \gamma \cap \mathbb{Z}_+^n} a_v x^v$.

If f is convenient and non-degenerate at infinity, then by a result of Broughton [2] it is tame at infinity. In this case, the monodromy zeta function $\zeta_f^{\infty}(t)$ has the following beautiful expression. For each non-empty subset $S \subset \{1, 2, \dots, n\}$, let $\{\gamma_1^S, \gamma_2^S, \dots, \gamma_{n(S)}^S\}$ be the $(\sharp S - 1)$-dimensional faces of $\Gamma_{\infty}^S(f)$ such that $0 \notin \gamma_i^S$. For $1 \leq i \leq n(S)$, let $u_i^S \in (\mathbb{R}^S)^* \cap \mathbb{Z}^S$ be the unique non-zero primitive vector which takes its maximum in $\Gamma_{\infty}^S(f)$ exactly on γ_i^S and set

$$d_i^S := \max_{v \in \Gamma_{\infty}^S(f)} \langle u_i^S, v \rangle \in \mathbb{Z}_{>0}. \tag{3.5}$$

We call d_i^S the lattice distance of γ_i^S from the origin $0 \in \mathbb{R}^S$. For each face $\gamma_i^S \prec \Gamma_{\infty}^S(f)$, let $\mathbb{L}(\gamma_i^S)$ be the smallest affine linear subspace of $\mathbb{R}^n$ containing γ_i^S and $\mathrm{Vol}_{\mathbb{Z}}(\gamma_i^S) \in \mathbb{Z}_{>0}$ the normalized $(\sharp S - 1)$-dimensional volume (i.e. the $(\sharp S - 1)!$ times the usual volume) of γ_i^S with respect to the lattice $\mathbb{Z}^n \cap \mathbb{L}(\gamma_i^S)$.

Theorem 3.6 ([24], see also [28] for a slight generalization) *Assume that f is convenient and non-degenerate at infinity. Then we have*

$$\zeta_f^{\infty}(t) = \prod_{S \neq \emptyset} \zeta_{f,S}^{\infty}(t), \tag{3.6}$$

where for each non-empty subset $S \subset \{1, 2, \dots, n\}$ we set

$$\zeta_{f,S}^{\infty}(t) := \prod_{i=1}^{n(S)} (1 - t^{d_i^S})^{(-1)^{\sharp S - 1} \mathrm{Vol}_{\mathbb{Z}}(\gamma_i^S)}. \tag{3.7}$$

This theorem was first proved by Libgober-Sperber [24]. Here for the reader's convenience, we briefly recall our new proof in [28] which will be frequently used in this paper.

Proof. Let $j \colon \mathbb{C} \longrightarrow \mathbb{P}^1 = \mathbb{C} \sqcup \{\infty\}$ be the compactification and set $\mathcal{F} := j_!(Rf_!\mathbb{C}_{\mathbb{C}^n}) \in \mathbf{D}_c^b(\mathbb{P}^1)$. Take a local coordinate h of $\mathbb{P}^1$ in a neighborhood of $\infty \in \mathbb{P}^1$ such that $\infty = \{h = 0\}$. Then by the isomorphism $H_j(f^{-1}(R); \mathbb{C}) \simeq H_c^{2n-2-j}(f^{-1}(R); \mathbb{C})$ we see that

$$\zeta_f^{\infty}(t) = \zeta_{h,\infty}(\mathcal{F})(t) \in \mathbb{C}(t)^*. \tag{3.8}$$

Now let us consider $\mathbb{C}^n$ as a toric variety associated with the fan Σ_0 in $\mathbb{R}^n$ formed by the all faces of the first quadrant $\mathbb{R}_+^n := (\mathbb{R}_{\geq 0})^n \subset \mathbb{R}^n$. Let $T \simeq (\mathbb{C}^*)^n$ be the open dense torus in it. Then by the convenience of f, Σ_0 is a subfan of the dual fan Σ_1 of $\Gamma_{\infty}(f)$ and we can construct a smooth subdivision Σ of Σ_1 without subdividing the cones in Σ_0 (see e.g. [34, Lemma (2.6), Chapter II, page 99]). This implies that the toric variety X_{Σ} associated with Σ is a smooth compactification of $\mathbb{C}^n$. Recall that T acts on X_{Σ} and the T-orbits are parametrized by the cones in Σ. For a cone $\sigma \in \Sigma$ denote by $T_{\sigma} \simeq (\mathbb{C}^*)^{n - \dim \sigma}$ the corresponding T-orbit. We have also natural affine open subsets $\mathbb{C}^n(\sigma) \simeq \mathbb{C}^n$ of X_{Σ} associated to n-dimensional cones σ in Σ. Let σ be an n-dimensional cone in Σ and $\{w_1, \dots, w_n\} \subset \mathbb{Z}^n$ the set of the primitive vectors on the edges of σ. Then there exists an affine open subset $\mathbb{C}^n(\sigma)$ of X_{Σ} such that $\mathbb{C}^n(\sigma) \simeq \mathbb{C}_y^n$ and f has the following form on it:

$$f(y) = \sum_{v \in \mathbb{Z}_+^n} a_v y_1^{\langle w_1, v \rangle} \cdots y_n^{\langle w_n, v \rangle} = y_1^{b_1} \cdots y_n^{b_n} \times f_{\sigma}(y), \tag{3.9}$$

where we set $f = \sum_{v \in \mathbb{Z}_+^n} a_v x^v$,

$$b_i = \min_{v \in \Gamma_\infty(f)} \langle w_i, v \rangle \leq 0 \qquad (i = 1, 2, \ldots, n) \tag{3.10}$$

and $f_\sigma(y)$ is a polynomial on $\mathbb{C}^n(\sigma) \simeq \mathbb{C}_y^n$. In $\mathbb{C}^n(\sigma) \simeq \mathbb{C}_y^n$ the hypersurface $Z := \overline{f^{-1}(0)} \subset X_\Sigma$ is explicitly written as $\{y \in \mathbb{C}^n(\sigma) \mid f_\sigma(y) = 0\}$. The variety X_Σ is covered by such affine open subsets. Let τ be a d-dimensional face of the n-dimensional cone $\sigma \in \Sigma$. For simplicity, assume that $w_1, \ldots, w_d$ generate τ. Then in the affine chart $\mathbb{C}^n(\sigma) \simeq \mathbb{C}_y^n$ the T-orbit T_τ associated to τ is explicitly defined by

$$T_\tau = \{(y_1, \ldots, y_n) \in \mathbb{C}^n(\sigma) \mid y_1 = \cdots = y_d = 0,\ y_{d+1}, \ldots, y_n \neq 0\} \simeq (\mathbb{C}^*)^{n-d}.$$

Hence we have

$$X_\Sigma = \bigcup_{\dim \sigma = n} \mathbb{C}^n(\sigma) = \bigsqcup_{\tau \in \Sigma} T_\tau. \tag{3.11}$$

Now f was extended to a meromorphic function $\widetilde{f}$ on X_Σ, but $\widetilde{f}$ has still points of indeterminacy. From now on, we will eliminate such points by blowing up X_Σ. For a cone σ in Σ by taking a non-zero vector u in the relative interior $\mathrm{rel.int}(\sigma)$ of σ we define a face $\gamma(\sigma)$ of $\Gamma_\infty(f)$ by

$$\gamma(\sigma) = \left\{ v \in \Gamma_\infty(f) \;\middle|\; \langle u, v \rangle = \min_{w \in \Gamma_\infty(f)} \langle u, w \rangle \right\}. \tag{3.12}$$

This face $\gamma(\sigma)$ does not depend on the choice of $u \in \mathrm{rel.int}(\sigma)$ and is called the supporting face of σ in $\Gamma_\infty(f)$. Following [24], we say that a T-orbit T_σ in X_Σ is at infinity if $0 \notin \gamma(\sigma)$. In our situation (i.e. f is convenient), this is equivalent to the condition $\sigma \not\subset \mathbb{R}_+^n$. We can easily see that $\widetilde{f}$ has poles on the union of T-orbits at infinity. Let $\rho_1, \rho_2, \ldots, \rho_m$ be the 1-dimensional cones in Σ such that $\rho_i \not\subset \mathbb{R}_+^n$ and set $T_i = T_{\rho_i}$. Then $T_1, T_2, \ldots, T_m$ are the $(n-1)$-dimensional T-orbits at infinity in X_Σ. For any $i = 1, 2, \ldots, m$ the toric divisor $D_i := \overline{T_i}$ is a smooth hypersurface in X_Σ and the poles of $\widetilde{f}$ are contained in $D_1 \cup \cdots \cup D_m$. Moreover by the non-degeneracy at infinity of f, the hypersurface $Z = \overline{f^{-1}(0)}$ in X_Σ intersects $D_I := \bigcap_{i \in I} D_i$ transversally for any non-empty subset $I \subset \{1, 2, \ldots, m\}$. At such intersection points, $\widetilde{f}$ has indeterminacy. Furthermore we denote the (unique non-zero) primitive vector in $\rho_i \cap \mathbb{Z}^n$ by u_i. Then the order $a_i > 0$ of the pole of $\widetilde{f}$ along D_i is given by

$$a_i = -\min_{v \in \Gamma_\infty(f)} \langle u_i, v \rangle. \tag{3.13}$$

Now, in order to eliminate the indeterminacy of the meromorphic function $\widetilde{f}$ on X_Σ, we first consider the blow-up $\pi_1 \colon X_\Sigma^{(1)} \longrightarrow X_\Sigma$ of X_Σ along the $(n-2)$-dimensional smooth subvariety $D_1 \cap Z$. Then the indeterminacy of the pull-back $\widetilde{f} \circ \pi_1$ of $\widetilde{f}$ to $X_\Sigma^{(1)}$ is improved. If $\widetilde{f} \circ \pi_1$ still has points of indeterminacy on the intersection of the exceptional divisor E_1 of π_1 and the proper transform $Z^{(1)}$ of Z, we construct the blow-up $\pi_2 \colon X_\Sigma^{(2)} \longrightarrow X_\Sigma^{(1)}$ of $X_\Sigma^{(1)}$ along $E_1 \cap Z^{(1)}$. By repeating this procedure a_1 times, we obtain a tower of blow-ups

$$X_\Sigma^{(a_1)} \underset{\pi_{a_1}}{\longrightarrow} \cdots\cdots \underset{\pi_2}{\longrightarrow} X_\Sigma^{(1)} \underset{\pi_1}{\longrightarrow} X_\Sigma. \tag{3.14}$$

Then the pull-back of $\widetilde{f}$ to $X_\Sigma^{(a_1)}$ has no indeterminacy over T_1 (see the figures below).

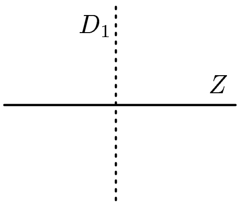

Figure 1

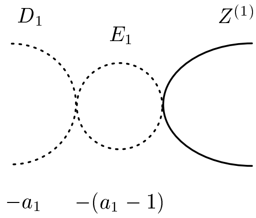

Figure 2

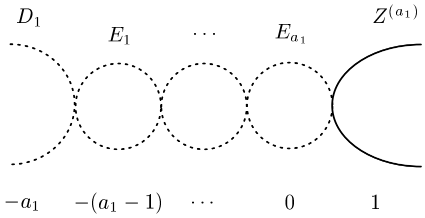

Figure 3

Next we apply this construction to the proper transforms of D_2 and Z in $X_\Sigma^{(a_1)}$. Then we obtain also a tower of blow-ups

$$X_\Sigma^{(a_1)(a_2)} \longrightarrow \cdots\cdots \longrightarrow X_\Sigma^{(a_1)(1)} \longrightarrow X_\Sigma^{(a_1)} \tag{3.15}$$

and the indeterminacy of the pull-back of $\widetilde{f}$ to $X_\Sigma^{(a_1)(a_2)}$ is eliminated over $T_1 \sqcup T_2$. By applying the same construction to (the proper transforms of) $D_3, D_4, \dots, D_m$, we finally obtain a birational morphism $\pi \colon \widetilde{X_\Sigma} \longrightarrow X_\Sigma$ such that $g := \widetilde{f} \circ \pi$ has no point of indeterminacy on the whole $\widetilde{X_\Sigma}$. Note that the smooth compactification $\widetilde{X_\Sigma}$ of $\mathbb{C}^n$ thus obtained is not a toric variety any more. We shall explain the geometry of $\widetilde{X_\Sigma}$ more precisely in the proof of Lemma 4.9. In particular, we will see that the union of the exceptional divisors of $\pi \colon \widetilde{X_\Sigma} \longrightarrow X_\Sigma$ and the proper transforms of $D_1, \dots, D_m$ in $\widetilde{X_\Sigma}$ is normal crossing. Finally we get a commutative diagram of holomorphic maps

$$\begin{tikzcd} \mathbb{C}^n \arrow[r, hook, "\iota"] \arrow[d, "f"'] & \widetilde{X_\Sigma} \arrow[d, "g"] \\ \mathbb{C} \arrow[r, hook, "j"] & \mathbb{P}^1, \end{tikzcd} \tag{3.16}$$

where g is proper. Therefore we obtain an isomorphism $\mathcal{F} = j_!(Rf_!\mathbb{C}_{\mathbb{C}^n}) \simeq Rg_*(\iota_!\mathbb{C}_{\mathbb{C}^n})$ in $\mathbf{D}_c^b(\mathbb{P}^1)$. Let us apply Proposition 2.3 to the proper morphism $g \colon \widetilde{X_\Sigma} \longrightarrow \mathbb{P}^1$. Then by calculating the monodromy zeta function of $\psi_{h\circ g}(\iota_!\mathbb{C}_{\mathbb{C}^n})$ at each point of $(h \circ g)^{-1}(0) = g^{-1}(\infty) \subset \widetilde{X_\Sigma}$, we can calculate $\zeta_{h,\infty}(\mathcal{F})(t)$ with the help of Bernstein-Khovanskii-Kushnirenko's theorem (see [20] etc.). This completes the proof. $\square$

4 Motivic Milnor fibers at infinity

In this section, following Denef-Loeser [6] and [7] we introduce motivic reincarnations of global (Milnor) fibers of polynomial maps and give a general formula for the nilpotent parts (i.e. the numbers of Jordan blocks of arbitrary sizes) in their monodromies at infinity. Namely, we formulate a global analogue of the results in [6] and [7]. Let $f \colon \mathbb{C}^n \longrightarrow \mathbb{C}$ be a polynomial map. We take a smooth compactification X of $\mathbb{C}^n$. Then by eliminating the points of indeterminacy of the meromorphic extension of f to X we obtain a commutative diagram

$$\begin{tikzcd} \mathbb{C}^n \arrow[r, hook, "\iota"] \arrow[d, "f"'] & \widetilde{X} \arrow[d, "g"] \\ \mathbb{C} \arrow[r, hook, "j"] & \mathbb{P}^1 \end{tikzcd} \tag{4.1}$$

such that g is a proper holomorphic map and $\widetilde{X} \setminus \mathbb{C}^n$, $Y := g^{-1}(\infty)$ are normal crossing divisors in $\widetilde{X}$. Take a local coordinate h of $\mathbb{P}^1$ in a neighborhood of $\infty \in \mathbb{P}^1$ such that

$\infty = \{h = 0\}$ and set $\widetilde{g} = h \circ g$. Note that $\widetilde{g}$ is a holomorphic function defined on a neighborhood of the closed subvariety $Y = \widetilde{g}^{-1}(0) = g^{-1}(\infty) \subset \widetilde{X} \setminus \mathbb{C}^n$ of $\widetilde{X}$. Then for $R \gg 0$ we have

$$H_c^j(f^{-1}(R); \mathbb{C}) \simeq H^j(Y; \psi_{\widetilde{g}}(\iota_! \mathbb{C}_{\mathbb{C}^n})). \tag{4.2}$$

Let us define an open subset Ω of $\widetilde{X}$ by

$$\Omega = \mathrm{Int}(\iota(\mathbb{C}^n) \sqcup Y) \tag{4.3}$$

and set $U = \Omega \cap Y$. Then U (resp. the complement of Ω in $\widetilde{X}$) is a normal crossing divisor in Ω (resp. $\widetilde{X}$). Hence we can easily prove the isomorphisms

$$H^j(Y; \psi_{\widetilde{g}}(\iota_! \mathbb{C}_{\mathbb{C}^n})) \simeq H^j(Y; \psi_{\widetilde{g}}(\iota'_! \mathbb{C}_{\Omega})) \simeq H_c^j(U; \psi_{\widetilde{g}}(\mathbb{C}_{\widetilde{X}})), \tag{4.4}$$

where $\iota' \colon \Omega \hookrightarrow \widetilde{X}$ is the inclusion. Now let $E_1, E_2, \dots, E_k$ be the irreducible components of the normal crossing divisor $U = \Omega \cap Y$ in $\Omega \subset \widetilde{X}$. For each $1 \leq i \leq k$, let $b_i > 0$ be the order of the zero of $\widetilde{g}$ along E_i. For a non-empty subset $I \subset \{1, 2, \dots, k\}$, let us set

$$E_I = \bigcap_{i \in I} E_i, \qquad E_I^\circ = E_I \setminus \bigcup_{i \notin I} E_i \tag{4.5}$$

and $d_I = \gcd(b_i)_{i \in I} > 0$. Then, as in [7, Section 3.3], we can construct an unramified Galois covering $\widetilde{E_I^\circ} \longrightarrow E_I^\circ$ of E_I° as follows. First, for a point $p \in E_I^\circ$ we take an affine open neighborhood $W \subset \Omega \setminus (\cup_{i \notin I} E_i)$ of p on which there exists a local coordinate system (i.e. a regular sequence) $\xi_1, \xi_2, \dots, \xi_n$ such that $E_i \cap W = \{\xi_i = 0\}$ for any $i \in I$. Then on W we have $\widetilde{g} = \widetilde{g_{1,W}} (\widetilde{g_{2,W}})^{d_I}$, where we set $\widetilde{g_{1,W}} = \widetilde{g} \prod_{i \in I} \xi_i^{-b_i}$ and $\widetilde{g_{2,W}} = \prod_{i \in I} \xi_i^{\frac{b_i}{d_I}}$. Note that $\widetilde{g_{1,W}}$ is a unit on W and $\widetilde{g_{2,W}} \colon W \longrightarrow \mathbb{C}$ is a regular function. It is easy to see that E_I° is covered by such affine open subsets W of $\Omega \setminus (\cup_{i \notin I} E_i)$. Then as in [7, Section 3.3] by gluing the varieties

$$\widetilde{E_{I,W}^\circ} = \{(t, z) \in \mathbb{C}^* \times (E_I^\circ \cap W) \mid t^{d_I} = (\widetilde{g_{1,W}})^{-1}(z)\} \tag{4.6}$$

together in the following way, we obtain the variety $\widetilde{E_I^\circ}$ over E_I°. If W' is another such open subset and $\widetilde{g} = \widetilde{g_{1,W'}} (\widetilde{g_{2,W'}})^{d_I}$ is the decomposition of $\widetilde{g}$ on it, we patch $\widetilde{E_{I,W}^\circ}$ and $\widetilde{E_{I,W'}^\circ}$ by the morphism $(t, z) \longmapsto (\widetilde{g_{2,W'}}(z) (\widetilde{g_{2,W}})^{-1}(z) \cdot t, z)$ defined over $W \cap W'$.

Remark 4.1 Let $N > 0$ be the least common multiple of $b_1, \dots, b_k$. As in Steenbrink [44], by taking the normalization of the base change of $\widetilde{g} \colon \Omega \longrightarrow \mathbb{C}$ by the N-th power map $\mathbb{C} \longrightarrow \mathbb{C}$ we obtain a morphism $\Omega' \longrightarrow \Omega$. Then it is well-known that the variety $\widetilde{E_I^\circ}$ is obtained as a connected component of the inverse image of E_I° by $\Omega' \longrightarrow \Omega$ (see Looijenga [25]). Moreover Looijenga [25, Lemma 5.3] proved that $\widetilde{E_I^\circ} \longrightarrow E_I^\circ$ is the Stein factorization of a fiber bundle over E_I°, which shows why the term $(1 - \mathbb{L})^{\sharp I - 1}$ appears in (4.7).

Now for $d \in \mathbb{Z}_{>0}$, let $\mu_d \simeq \mathbb{Z}/\mathbb{Z}d$ be the multiplicative group consisting of the d-roots in $\mathbb{C}$. We denote by $\hat{\mu}$ the projective limit $\varprojlim_d \mu_d$ of the projective system $\{\mu_i\}_{i \geq 1}$ with morphisms $\mu_{id} \longrightarrow \mu_i$ given by $t \longmapsto t^d$. Then the unramified Galois covering $\widetilde{E_I^\circ}$ of E_I°

admits a natural μ_{d_I}-action defined by assigning the automorphism $(t,z) \longmapsto (\zeta_{d_I}t, z)$ of $\widetilde{E_I^\circ}$ to the generator $\zeta_{d_I} := \exp(2\pi\sqrt{-1}/d_I) \in \mu_{d_I}$. Namely the variety $\widetilde{E_I^\circ}$ is equipped with a good $\hat{\mu}$-action in the sense of [7, Section 2.4]. Following the notations in [7], denote by $\mathcal{M}_{\mathbb{C}}^{\hat{\mu}}$ the ring obtained from the Grothendieck ring $\mathrm{K}_0^{\hat{\mu}}(\mathrm{Var}_{\mathbb{C}})$ of varieties over $\mathbb{C}$ with good $\hat{\mu}$-actions by inverting the Lefschetz motive $\mathbb{L} \simeq \mathbb{C} \in \mathrm{K}_0^{\hat{\mu}}(\mathrm{Var}_{\mathbb{C}})$. Recall that $\mathbb{L} \in \mathrm{K}_0^{\hat{\mu}}(\mathrm{Var}_{\mathbb{C}})$ is endowed with the trivial action of $\hat{\mu}$.

Definition 4.2 We define the motivic Milnor fiber at infinity $\mathcal{S}_f^\infty$ of the polynomial map $f\colon \mathbb{C}^n \longrightarrow \mathbb{C}$ by

$$\mathcal{S}_f^\infty = \sum_{I \neq \emptyset} (1-\mathbb{L})^{\sharp I - 1}[\widetilde{E_I^\circ}] \in \mathcal{M}_{\mathbb{C}}^{\hat{\mu}}. \tag{4.7}$$

Remark 4.3 By Guibert-Loeser-Merle [17, Theorem 3.9], the motivic Milnor fiber at infinity $\mathcal{S}_f^\infty$ of f does not depend on the compactification X of $\mathbb{C}^n$. This fact was informed to us by Schürmann (a private communication) and Raibaut [35].

As in [7, Section 3.1.2 and 3.1.3], we denote by $\mathrm{HS}^{\mathrm{mon}}$ the abelian category of Hodge structures with a quasi-unipotent endomorphism. Then, to the object $\psi_h(j_! Rf_! \mathbb{C}_{\mathbb{C}^n}) \in \mathbf{D}_c^b(\{\infty\})$ and the semisimple part of the monodromy automorphism acting on it, we can associate an element

$$[H_f^\infty] \in \mathrm{K}_0(\mathrm{HS}^{\mathrm{mon}}) \tag{4.8}$$

in an obvious way. Similarly, to $\psi_h(Rj_* Rf_* \mathbb{C}_{\mathbb{C}^n}) \in \mathbf{D}_c^b(\{\infty\})$ we associate an element

$$[G_f^\infty] \in \mathrm{K}_0(\mathrm{HS}^{\mathrm{mon}}). \tag{4.9}$$

According to a deep result [38, Theorem 13.1] of Sabbah, if f is tame at infinity then the weights of the element $[G_f^\infty]$ are defined by the monodromy filtration up to some Tate twists (see also [40] and [42]). This implies that for the calculation of the monodromy at infinity $\Phi_{n-1}^\infty\colon H^{n-1}(f^{-1}(R);\mathbb{C}) \xrightarrow{\sim} H^{n-1}(f^{-1}(R);\mathbb{C})$ $(R \gg 0)$ of f it suffices to calculate $[H_f^\infty] \in \mathrm{K}_0(\mathrm{HS}^{\mathrm{mon}})$ which is the dual of $[G_f^\infty]$.

To describe the element $[H_f^\infty] \in \mathrm{K}_0(\mathrm{HS}^{\mathrm{mon}})$ in terms of $\mathcal{S}_f^\infty \in \mathcal{M}_{\mathbb{C}}^{\hat{\mu}}$, let

$$\chi_h\colon \mathcal{M}_{\mathbb{C}}^{\hat{\mu}} \longrightarrow \mathrm{K}_0(\mathrm{HS}^{\mathrm{mon}}) \tag{4.10}$$

be the Hodge characteristic morphism defined in [7] which associates to a variety Z with a good μ_d-action the Hodge structure

$$\chi_h([Z]) = \sum_{j \in \mathbb{Z}} (-1)^j [H_c^j(Z;\mathbb{Q})] \in \mathrm{K}_0(\mathrm{HS}^{\mathrm{mon}}) \tag{4.11}$$

with the actions induced by the one $z \longmapsto \exp(2\pi\sqrt{-1}/d)z$ $(z \in Z)$ on Z. Then by applying the proof of [6, Theorem 4.2.1] to our situation (4.2) and (4.4), we obtain the following result.

Theorem 4.4 *In the Grothendieck group* $\mathrm{K}_0(\mathrm{HS}^{\mathrm{mon}})$, *we have*

$$[H_f^\infty] = \chi_h(\mathcal{S}_f^\infty). \tag{4.12}$$

On the other hands, the results in [36] and [38] imply the following symmetry of the weights of the element $[H_f^\infty] \in \mathrm{K}_0(\mathrm{HS}^{\mathrm{mon}})$ when f is tame at infinity (see Appendix for the details). Recall that if f is tame at infinity we have $H_c^j(f^{-1}(R);\mathbb{C}) = 0$ $(R \gg 0)$ for $j \neq n-1, 2n-2$ and $H_c^{2n-2}(f^{-1}(R);\mathbb{C}) \simeq [H^0(f^{-1}(R);\mathbb{C})]^* \simeq \mathbb{C}$. For an element $[V] \in \mathrm{K}_0(\mathrm{HS}^{\mathrm{mon}})$, $V \in \mathrm{HS}^{\mathrm{mon}}$ with a quasi-unipotent endomorphism $\Theta \colon V \overset{\sim}{\longrightarrow} V$, $p, q \geq 0$ and $\lambda \in \mathbb{C}$ denote by $e^{p,q}([V])_\lambda$ the dimension of the λ-eigenspace of the morphism $V^{p,q} \overset{\sim}{\longrightarrow} V^{p,q}$ induced by Θ on the (p,q)-part $V^{p,q}$ of V.

Theorem 4.5 (Sabbah [36] and [38]) *Assume that f is tame at infinity. Then*

(i) *Let $\lambda \in \mathbb{C}^* \setminus \{1\}$. Then we have $e^{p,q}([H_f^\infty])_\lambda = 0$ for $(p,q) \notin [0, n-1] \times [0, n-1]$. Moreover for $(p,q) \in [0, n-1] \times [0, n-1]$ we have*

$$e^{p,q}([H_f^\infty])_\lambda = e^{n-1-q, n-1-p}([H_f^\infty])_\lambda. \tag{4.13}$$

(ii) *We have $e^{p,q}([H_f^\infty])_1 = 0$ for $(p,q) \notin \{(n-1, n-1)\} \sqcup ([0, n-2] \times [0, n-2])$ and $e^{n-1,n-1}([H_f^\infty])_1 = 1$. Moreover for $(p,q) \in [0, n-2] \times [0, n-2]$ we have*

$$e^{p,q}([H_f^\infty])_1 = e^{n-2-q, n-2-p}([H_f^\infty])_1. \tag{4.14}$$

Using our results in Section 5, we can check the above symmetry by explicitly calculating $\chi_h(\mathcal{S}_f^\infty)$ for small n's. Since the weights of $[G_f^\infty] \in \mathrm{K}_0(\mathrm{HS}^{\mathrm{mon}})$ are defined by the monodromy filtration and $[G_f^\infty]$ is the dual of $[H_f^\infty]$ up to some Tate twist, we obtain the following result.

Theorem 4.6 *Assume that f is tame at infinity. Then*

(i) *Let $\lambda \in \mathbb{C}^* \setminus \{1\}$ and $k \geq 1$. Then the number of the Jordan blocks for the eigenvalue λ with sizes $\geq k$ in $\Phi_{n-1}^\infty \colon H^{n-1}(f^{-1}(R);\mathbb{C}) \overset{\sim}{\longrightarrow} H^{n-1}(f^{-1}(R);\mathbb{C})$ $(R \gg 0)$ is equal to*

$$(-1)^{n-1} \sum_{p+q = n-2+k, n-1+k} e^{p,q}(\chi_h(\mathcal{S}_f^\infty))_\lambda. \tag{4.15}$$

(ii) *For $k \geq 1$, the number of the Jordan blocks for the eigenvalue 1 with sizes $\geq k$ in Φ_{n-1}^∞ is equal to*

$$(-1)^{n-1} \sum_{p+q = n-2-k, n-1-k} e^{p,q}(\chi_h(\mathcal{S}_f^\infty))_1. \tag{4.16}$$

By using Newton polyhedrons at infinity, we can rewrite the result of Theorem 4.4 more neatly as follows. Let $f \in \mathbb{C}[x_1, \dots, x_n]$ be a convenient polynomial. Assume that f is non-degenerate at infinity. Then f is tame at infinity and it suffices to calculate Φ_j^∞ only for $j = n-1$. From now on, we will freely use the notations in the proof of Theorem 3.6. For example, $\rho_1, \dots, \rho_m$ are the 1-dimensional cones in the smooth fan Σ such that $\rho_i \not\subset \mathbb{R}_+^n$. We call these cones the rays at infinity. Each ray ρ_i at infinity corresponds to the toric divisor D_i in X_Σ and the divisor $D := D_1 \cup \cdots \cup D_m = X_\Sigma \setminus \mathbb{C}^n$ in X_Σ is normal crossing. We denote by $a_i > 0$ the order of the poles of f along D_i. By eliminating

the points of indeterminacy of the meromorphic extension of f to X_Σ we constructed the commutative diagram:

$$\begin{tikzcd} \mathbb{C}^n \arrow[r, hook, "\iota"] \arrow[d, "f"'] & \widetilde{X_\Sigma} \arrow[d, "g"] \\ \mathbb{C} \arrow[r, hook, "j"] & \mathbb{P}^1. \end{tikzcd} \tag{4.17}$$

Recall that in the construction of $\widetilde{X_\Sigma}$ we first construct a tower of blow-ups over $D_1 \cap \overline{f^{-1}(0)}$ and next apply the same operation to the remaining divisors $D_2, \dots, D_m$ (in this order). See the proof of Theorem 3.6 and Lemma 4.9 for the details. Take a local coordinate h of $\mathbb{P}^1$ in a neighborhood of $\infty \in \mathbb{P}^1$ such that $\infty = \{h = 0\}$ and set $\widetilde{g} = h \circ g$, $Y = \widetilde{g}^{-1}(0) = g^{-1}(\infty) \subset \widetilde{X_\Sigma}$ and $\Omega = \mathrm{Int}(\iota(\mathbb{C}^n) \sqcup Y)$. For simplicity, let us set $\widetilde{g} = \frac{1}{f}$. Then the divisor $U = Y \cap \Omega$ in Ω contains not only the proper transforms $D_1', \dots, D_m'$ of $D_1, \dots, D_m$ in $\widetilde{X_\Sigma}$ but also the exceptional divisors of the blow-up: $\widetilde{X_\Sigma} \longrightarrow X_\Sigma$. From now on, we will show that these exceptional divisors are not necessary to compute the monodromy at infinity of $f \colon \mathbb{C}^n \longrightarrow \mathbb{C}$ by Theorem 4.4. For each non-empty subset $I \subset \{1, 2, \dots, m\}$, set $D_I = \bigcap_{i \in I} D_i$,

$$D_I^\circ = D_I \setminus \left\{ \left(\bigcup_{i \notin I} D_i \right) \cup \overline{f^{-1}(0)} \right\} \subset X_\Sigma \tag{4.18}$$

and $d_I = \gcd(a_i)_{i \in I} > 0$. Then the function $\widetilde{g} = \frac{1}{f}$ is regular on D_I° and we can decompose it as $\frac{1}{f} = \widetilde{g_1}(\widetilde{g_2})^{d_I}$ globally on a Zariski open neighborhood W of D_I° in X_Σ, where $\widetilde{g_1}$ is a unit on W and $\widetilde{g_2} \colon W \longrightarrow \mathbb{C}$ is regular. Therefore we can construct an unramified Galois covering $\widetilde{D_I^\circ}$ of D_I° with a natural μ_{d_I}-action as in (4.6). Let $[\widetilde{D_I^\circ}]$ be the element of the ring $\mathcal{M}_{\mathbb{C}}^{\hat{\mu}}$ which corresponds to $\widetilde{D_I^\circ}$.

Theorem 4.7 *In the situation as above, we have the equality*

$$\chi_h\left(\mathcal{S}_f^\infty\right) = \sum_{I \neq \emptyset} \chi_h\left((1 - \mathbb{L})^{\sharp I - 1}[\widetilde{D_I^\circ}]\right) \tag{4.19}$$

in the Grothendieck group $\mathrm{K}_0(\mathrm{HS}^{\mathrm{mon}})$.

Proof. First, we prove the assertion for $n = 2$. In this case, we number the rays at infinity in Σ in the clockwise direction as in the figure below.

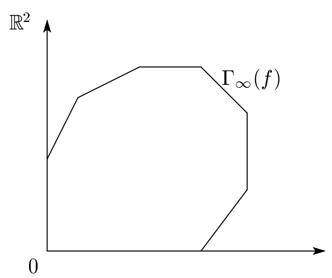

Figure 4

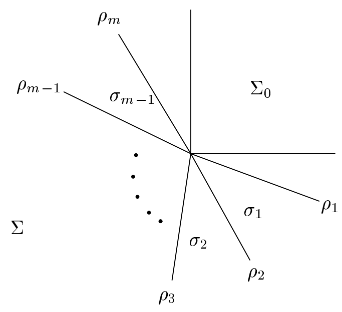

Figure 5

Let $\sigma_i = \mathbb{R}_+\rho_i + \mathbb{R}_+\rho_{i+1}$ $(1 \leq i \leq m-1)$ be the 2-dimensional cone in Σ between ρ_i and ρ_{i+1}. Then the cone σ_i corresponds to an affine open subset $\mathbb{C}^2(\sigma_i) \simeq \mathbb{C}^2_{\xi,\eta}$ of X_Σ on

which the meromorphic extension of $\frac{1}{f}$ to X_Σ has the form

$$\left(\frac{1}{f}\right)(\xi,\eta)=\frac{\xi^{a_i}\eta^{a_{i+1}}}{f_{\sigma_i}(\xi,\eta)}, \tag{4.20}$$

where f_{σ_i} is a polynomial of ξ and η. In this situation, we have $D_i \cap \mathbb{C}^2(\sigma_i)=\{\xi=0\}$ and $D_{i+1}\cap \mathbb{C}^2(\sigma_i)=\{\eta=0\}$. Moreover by the non-degeneracy at infinity of $f(x,y)\in\mathbb{C}[x,y]$ we have $f_{\sigma_i}(0,0)\neq 0$ and the algebraic curve $f_{\sigma_i}^{-1}(0)=\{(\xi,\eta)\mid f_{\sigma_i}(\xi,\eta)=0\}$ intersects $D_i\cap\mathbb{C}^2(\sigma_i)$ and $D_{i+1}\cap\mathbb{C}^2(\sigma_i)$ transversally.

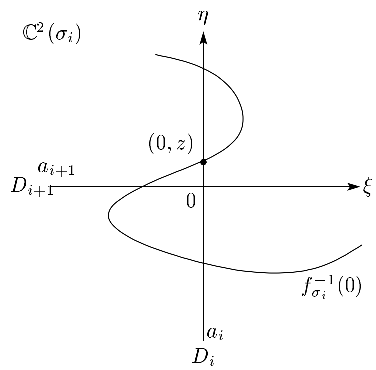

Figure 6

Let $(0,z)$, $z\neq 0$ be a point of $D_i\cap f_{\sigma_i}^{-1}(0)$. In constructing the variety $\widetilde{X_\Sigma}$, we constructed a tower of blow-ups over this point $(0,z)$ as in the figure below.

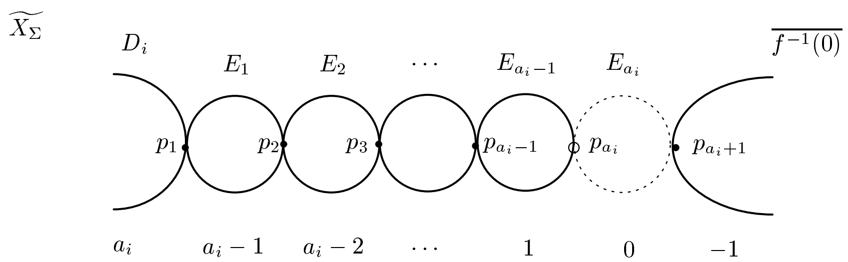

Figure 7

Here we have $E_j\simeq\mathbb{P}^1$ and the function $\frac{1}{f}=\widetilde{g}$ has the zero of order a_i-j along the exceptional divisor E_j. The open set Ω is the complement of $E_{a_i}\simeq\mathbb{P}^1$ in this figure. For $1\leq j\leq a_i-1$, set $E_j^\circ := E_j\setminus\{p_j,p_{j+1}\}$ and let $\widetilde{E_j^\circ}$ be the unramified Galois covering of E_j° with a μ_{a_i-j}-action (in the construction of $\mathcal{S}_f^\infty$). The motivic Milnor fiber at infinity $\mathcal{S}_f^\infty$ also contains $(1-\mathbb{L})\cdot[p_j]\in\mathcal{M}_{\mathbb{C}}^{\hat{\mu}}$ with the trivial $\hat{\mu}$-action for $1\leq j\leq a_i-1$.

Lemma 4.8 *For $1\leq j\leq a_i-1$, we have*

$$\chi_h\left((1-\mathbb{L})\cdot[p_j]+[\widetilde{E_j^\circ}]\right)=0 \tag{4.21}$$

in the Grothendieck group $\mathrm{K}_0(\mathrm{HS}^{\mathrm{mon}})$.

Proof. First set $X = \mathbb{C}^2(\sigma_i) \setminus \{\eta = 0\}$ and $Y = \{(0, z)\} \subset X$ and consider the regular functions

$$f_1(\xi, \eta) = \xi, \qquad f_2(\xi, \eta) = \frac{f_{\sigma_i}(\xi, \eta)}{\eta^{a_{i+1}}} \tag{4.22}$$

on X. Then (in a neighborhood of Y) the blow-up $\widetilde{X_Y}$ of X along Y is isomorphic to the closure of the image of the morphism

$$X \setminus Y \longrightarrow X \times \mathbb{P}^1 \tag{4.23}$$

given by

$$(\xi, \eta) \longmapsto (\xi, \eta, (f_1(\xi, \eta) : f_2(\xi, \eta))). \tag{4.24}$$

Let $\pi \colon \widetilde{X_Y} \twoheadrightarrow X$ be the natural morphism and define an open subset W of $\widetilde{X_Y}$ by

$$W = \{(\xi, \eta, (1 : \alpha)) \in \widetilde{X_Y} \mid \alpha \in \mathbb{C}\}. \tag{4.25}$$

Then considering α as a regular function on W, on $W \subset \widetilde{X_Y}$ we have

$$\left(\frac{1}{f}\right) \circ \pi = \frac{(f_1 \circ \pi)^{a_i}}{(f_2 \circ \pi)} = \frac{(f_1 \circ \pi)^{a_i}}{\alpha(f_1 \circ \pi)} = \frac{(f_1 \circ \pi)^{a_i - 1}}{\alpha}. \tag{4.26}$$

Moreover in $W \subset \widetilde{X_Y}$ the exceptional divisor $E = \pi^{-1}(Y)$ ($\simeq E_1$) is defined by $E = \{f_1 \circ \pi = 0\}$. Therefore the unramified Galois covering $\widetilde{E_1^\circ}$ of $E_1^\circ \simeq \{\alpha \in \mathbb{C} \mid \alpha \neq 0\} \simeq \mathbb{C}^*$ is $\{(t, \alpha) \in (\mathbb{C}^*)^2 \mid t^{a_i - 1}\alpha^{-1} = 1\}$, which is isomorphic to $\mathbb{C}^*$ with an automorphism homotopic to the identity. Hence its Hodge characteristic $\chi_h\left([\widetilde{E_1^\circ}]\right) \in \mathrm{K}_0(\mathrm{HS}^{\mathrm{mon}})$ is isomorphic to $\chi_h(\mathbb{L} - 1)$. We thus obtain the equality

$$\chi_h\left((1 - \mathbb{L}) \cdot [p_1] + [\widetilde{E_1^\circ}]\right) = 0 \tag{4.27}$$

in $\mathrm{K}_0(\mathrm{HS}^{\mathrm{mon}})$. By repeating this argument, we can similarly prove the remaining assertions. ☐

Recall that the motivic Milnor fiber at infinity $\mathcal{S}_f^\infty$ is a sum of the unramified Galois coverings of some Zariski locally closed subvarieties of Ω. Then Lemma 4.8 above implies that the Hodge characteristics of the base changes of $\mathcal{S}_f^\infty$ to the exceptional divisors of $\Omega \longrightarrow X_\Sigma$ are zero in $\mathrm{K}_0(\mathrm{HS}^{\mathrm{mon}})$. In other words, for the calculation of $\chi_h\left(\mathcal{S}_f^\infty\right) \in \mathrm{K}_0(\mathrm{HS}^{\mathrm{mon}})$, we can forget the parts of $\chi_h\left(\mathcal{S}_f^\infty\right) \in \mathrm{K}_0(\mathrm{HS}^{\mathrm{mon}})$ coming from the exceptional divisors of $\Omega \longrightarrow X_\Sigma$. So the theorem was proved in the case $n = 2$.

From now on, we shall prove Theorem 4.7 in the case $n > 2$. Let $\pi_\Omega \colon \Omega \twoheadrightarrow X_\Sigma$ be the restriction of the morphism $\pi \colon \widetilde{X_\Sigma} \twoheadrightarrow X_\Sigma$ to Ω. For each non-empty subset $I \subset \{1, 2, \ldots, m\}$ set $D_I^* := D_I \setminus \left(\bigcup_{i \notin I} D_i\right)$. Then, to prove the theorem, it suffices to show that for any non-empty subset $I \subset \{1, 2, \ldots, m\}$ the Hodge characteristic of the base change of $\mathcal{S}_f^\infty$ to $\pi_\Omega^{-1}(D_I^* \cap \overline{f^{-1}(0)}) \subset \Omega$ is zero in $\mathrm{K}_0(\mathrm{HS}^{\mathrm{mon}})$. First, let us consider the case where $I = \{i\}$ for some $1 \leq i \leq m$. Then by the construction of $\widetilde{X_\Sigma}$ and Ω, the morphism π_Ω induces a fiber bundle

$$\pi_\Omega^{-1}\left(D_{\{i\}}^* \cap \overline{f^{-1}(0)}\right) \twoheadrightarrow D_{\{i\}}^* \cap \overline{f^{-1}(0)} \tag{4.28}$$

over $D^*_{\{i\}} \cap \overline{f^{-1}(0)}$ whose fiber is isomorphic to the (locally closed) curve $(E_1 \cup \cdots \cup E_{a_i-1}) \setminus \{p_{a_i}\}$ as in Figure 7. By the proof of Lemma 4.8, this fiber bundle is locally trivial with respect to the Zariski topology of $D^*_{\{i\}} \cap \overline{f^{-1}(0)}$, and the Hodge characteristic of the base change of $\mathcal{S}^\infty_f$ to $\pi_\Omega^{-1}(D^*_{\{i\}} \cap \overline{f^{-1}(0)}) \subset \Omega$ is zero in $\mathrm{K}_0(\mathrm{HS}^{\mathrm{mon}})$. Next, consider a general non-empty subset $I = \{i_1 < \cdots < i_k\}$ of $\{1, 2, \ldots, m\}$. Then we have the following lemma.

Lemma 4.9 *In the situation as above, the restriction of* π_Ω *to* $\pi_\Omega^{-1}(D^*_I \cap \overline{f^{-1}(0)})$*:*

$$\pi_\Omega^{-1}(D^*_I \cap \overline{f^{-1}(0)}) \longrightarrow D^*_I \cap \overline{f^{-1}(0)} \tag{4.29}$$

is a Zariski locally trivial bundle over $D^*_I \cap \overline{f^{-1}(0)}$ *whose fiber is isomorphic to the (locally closed) curve* $(E_1 \cup \cdots \cup E_{a_{i_1}-1}) \setminus \{p_{a_{i_1}}\}$ *in Figure 7 for* $i = i_1$.

Proof. First, let us consider the case where $\sharp I = 2$. Without loss of generality, we may assume that $I = \{1, 2\}$ as in the figure:

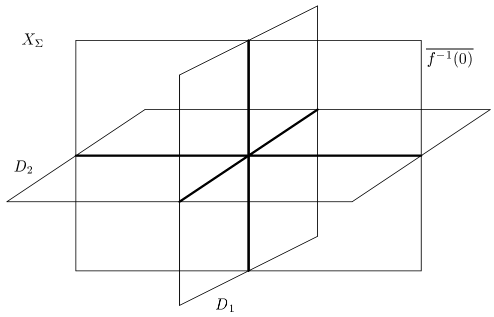

Figure 8

Although we consider the problem in a neighborhood of $D^*_I \subset X_\Sigma$, the total space of Figure 8 is denoted simply by X_Σ. In the construction of $\widetilde{X_\Sigma}$, we first construct a tower of blow-ups over $D_1 \cap \overline{f^{-1}(0)}$ as

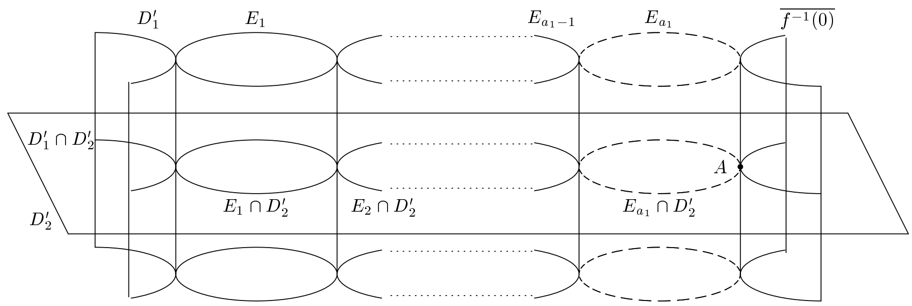

Figure 9

Here D'_1, D'_2 are the proper transforms of D_1, D_2 respectively. In Figure 9, the meromorphic function $\widetilde{g}$ still has points of indeterminacy on $D'_2 \cap \overline{f^{-1}(0)}$. Then we construct a tower of blow-ups over $D'_2 \cap \overline{f^{-1}(0)}$ as

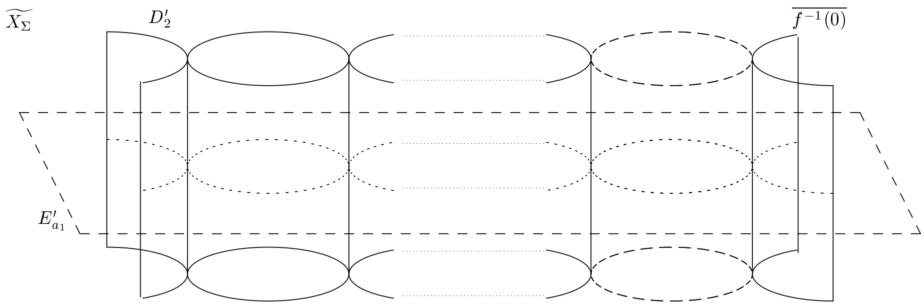

Figure 10

Since we are considering the problem only in a neighborhood of D_I^* and already finished the necessary blow-ups over D_I^*, the total space of Figure 10 is denoted simply by $\widetilde{X_\Sigma}$. In Figure 10, the open set $\Omega \subset \widetilde{X_\Sigma}$ is the complement of the union of two dotted divisors (E'_{a_1} is one of them). Moreover we see that the inverse image of the set A (in Figure 9) in $\widetilde{X_\Sigma}$ is contained in $\widetilde{X_\Sigma} \setminus \Omega$. This implies that $\pi_\Omega^{-1}(D_I^* \cap \overline{f^{-1}(0)})$ is the (locally closed) variety of the form:

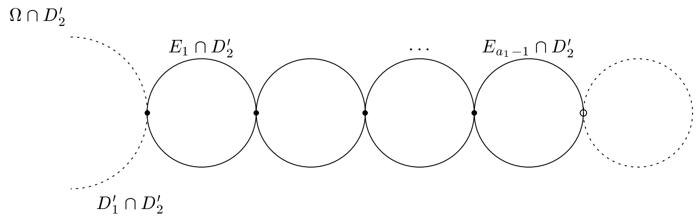

Figure 11

This completes the proof for the case $\sharp I = 2$. The general case can be proved similarly. $\square$

By the proof of Lemma 4.9 we see that $\pi_\Omega^{-1}(D_I^* \cap \overline{f^{-1}(0)})$ has a geometric structure as the figure below in $\Omega \setminus \pi_\Omega^{-1}\left(\bigcup_{i \notin I} D_i\right)$:

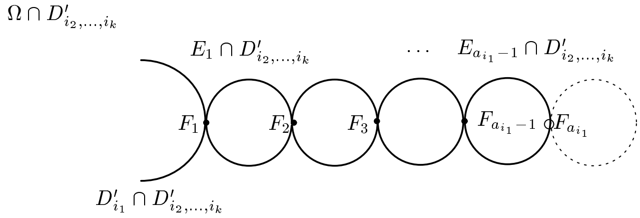

Figure 12

Here $E_1, \dots, E_{a_{i_1}}$ are the exceptional divisors in $\widetilde{X_\Sigma}$ constructed when we made a tower of blow-ups over $D_{i_1} \cap \overline{f^{-1}(0)}$ (We used essentially the condition $i_1 = \min\{i_1, \dots, i_k\}$. To simplify the notations, we denote $E_j \setminus \pi_\Omega^{-1}\left(\bigcup_{i \notin I} D_i\right)$ simply by E_j etc.). Moreover we set $D'_{i_2,\dots,i_k} := D'_{i_2} \cap \cdots \cap D'_{i_k}$ and $F_j := E_{j-1} \cap E_j \cap D'_{i_2,\dots,i_k}$. Note that $E_j \cap D'_{i_2,\dots,i_k}$ is a $\mathbb{P}^1$-bundle over $D_I^* \cap \overline{f^{-1}(0)}$. Let us set $(E_j \cap D'_{i_2,\dots,i_k})^\circ := (E_j \cap D'_{i_2,\dots,i_k}) \setminus (F_j \sqcup F_{j+1})$.

Then for each point p of $(E_j \cap D'_{i_2,\ldots,i_k})^\circ$ there exists a Zariski open neighborhood W of p in Ω and a local coordinate system $\xi_1, \xi_2, \ldots, \xi_n$ on it such that

$$D'_{i_2} = \{\xi_2 = 0\}, \ldots, D'_{i_k} = \{\xi_k = 0\}, \tag{4.30}$$

$$E_j \cap D'_{i_2,\ldots,i_k} = \{\xi_1 = \xi_2 = \cdots = \xi_k = 0\} \tag{4.31}$$

and the function $\frac{1}{f} = \widetilde{g}$ can be written in the form

$$\widetilde{g}(\xi_1, \ldots, \xi_n) = \xi_1^{a_{i_1}-j} \xi_2^{a_{i_2}} \cdots \xi_k^{a_{i_k}} \times (\text{ a unit on } W) \tag{4.32}$$

on W. Set $d_{I,j} = \gcd(a_{i_1} - j, a_{i_2}, \ldots, a_{i_k}) > 0$ $(1 \leq j \leq a_{i_1} - 1)$. Then the base change of $\mathcal{S}_f^\infty$ to $(E_j \cap D'_{i_2,\ldots,i_k})^\circ \subset \Omega$ is an unramified Galois covering $(\widetilde{E_j \cap D'_{i_2,\ldots,i_k}})^\circ$ of $(E_j \cap D'_{i_2,\ldots,i_k})^\circ$ with a natural $\mu_{d_{I,j}}$-action. Moreover by the proof of Lemma 4.8, we observe that $(\widetilde{E_j \cap D'_{i_2,\ldots,i_k}})^\circ$ is a (Zariski) locally trivial family over $D_I^* \cap \overline{f^{-1}(0)}$. By using this fact (and an analogue of [5, Proposition 1.6]), in the same way as the final part of the proof of Lemma 4.8 we obtain the equality

$$\chi_h \left((1 - \mathbb{L}) \cdot [F_j] + [(\widetilde{E_j \cap D'_{i_2,\ldots,i_k}})^\circ] \right) = 0 \tag{4.33}$$

in $\mathrm{K}_0(\mathrm{HS}^{\mathrm{mon}})$ for $1 \leq j \leq a_{i_1} - 1$, where $[F_j] \in \mathcal{M}_{\mathbb{C}}^{\hat{\mu}}$ is endowed with the trivial action of $\hat{\mu}$. This implies that the Hodge characteristic of the base change of $\mathcal{S}_f^\infty$ to $\pi_\Omega^{-1}(D_I^* \cap \overline{f^{-1}(0)}) \subset \Omega$ is zero in $\mathrm{K}_0(\mathrm{HS}^{\mathrm{mon}})$. In other words, the contribution to $\chi_h\left(\mathcal{S}_f^\infty\right) \in \mathrm{K}_0(\mathrm{HS}^{\mathrm{mon}})$ from the exceptional divisors of $\pi_\Omega \colon \Omega \longrightarrow X_\Sigma$ is zero. This completes the proof of Theorem 4.7. $\square$

Remark 4.10 It seems that the equality $\mathcal{S}_f^\infty = \sum_{I \neq \emptyset} (1 - \mathbb{L})^{\sharp I - 1} [\widetilde{D_I^\circ}]$ does not hold in $\mathcal{M}_{\mathbb{C}}^{\hat{\mu}}$. Indeed, we used a homotopy in the last part of the proof of Lemma 4.8 (and that of Theorem 4.7).

5 Combinatorial descriptions of monodromies at infinity

In this section, by rewriting Theorem 4.7 in terms of the Newton polyhedron at infinity $\Gamma_\infty(f)$ of f we prove some combinatorial formulas for the Jordan normal form of its monodromy at infinity Φ_{n-1}^∞. We inherit the situation and the notations in the last half of Section 4. Namely we assume that f is convenient and non-degenerate at infinity. Recall that $\rho_1, \rho_2, \ldots, \rho_m$ are 1-dimensional cones in the smooth fan Σ such that $\rho_i \not\subset \mathbb{R}_+^n$.

Definition 5.1 We say that $\gamma \prec \Gamma_\infty(f)$ is a face at infinity of $\Gamma_\infty(f)$ if $0 \notin \gamma$.

For a cone $\sigma \in \Sigma$ whose supporting face $\gamma(\sigma) \prec \Gamma_\infty(f)$ is at infinity ($\Longleftrightarrow \sigma \not\subset \mathbb{R}_+^n$) we set $I_\sigma = \{1 \leq i \leq m \mid \rho_i \prec \sigma\}$, $T_\sigma^\circ = T_\sigma \setminus \overline{f^{-1}(0)}$ and

$$\widetilde{T_\sigma^\circ} = \widetilde{D_{I_\sigma}^\circ} \cap (\mathbb{C}_t^* \times T_\sigma) \subset \mathbb{C}_t^* \times T_\sigma. \tag{5.1}$$

Then $\widetilde{T_\sigma^\circ}$ is a hypersurface in the algebraic torus $\mathbb{C}_t^* \times T_\sigma \simeq (\mathbb{C}^*)^{n-\dim\sigma+1}$ and a finite covering of T_σ°. Moreover for any non-empty subset $I \subset \{1, 2, \dots, m\}$ we have the decomposition:

$$\widetilde{D_I^\circ} = \bigsqcup_{\sigma\colon I_\sigma = I} \widetilde{T_\sigma^\circ}. \tag{5.2}$$

Therefore, for the calculation of $\chi_h([\widetilde{D_I^\circ}]) \in \mathrm{K}_0(\mathrm{HS}^{\mathrm{mon}})$ by the results in Section 2 we have to show that the hypersurfaces $\widetilde{T_\sigma^\circ} \subset \mathbb{C}_t^* \times T_\sigma \simeq (\mathbb{C}^*)^{n-\dim\sigma+1}$ are non-degenerate. Indeed, for such a cone $\sigma \in \Sigma$ let $\sigma_0 \in \Sigma$ be an n-dimensional cone such that $\sigma \prec \sigma_0$ and $\{w_1, w_2, \dots, w_n\} \subset \mathbb{Z}^n$ the set of the primitive vectors on the edges of σ_0. Set $\dim\sigma = k > 0$. Then we may assume that $w_1, \dots, w_k$ generate σ so that in the affine open subset $\mathbb{C}^n(\sigma_0) \simeq \mathbb{C}_y^n$ of X_Σ associated to σ_0 we have

$$T_\sigma = \{(y_1, \dots, y_n) \in \mathbb{C}^n(\sigma_0) \mid y_1 = \cdots = y_k = 0,\ y_{k+1}, \dots, y_n \neq 0\}. \tag{5.3}$$

On $\mathbb{C}^n(\sigma_0) \simeq \mathbb{C}_y^n$ the function $\widetilde{g} = \frac{1}{f}$ has the form:

$$\widetilde{g}(y) = y_1^{c_1} \cdots y_n^{c_n} \times \frac{1}{f_{\sigma_0}(y)}, \tag{5.4}$$

where we set

$$c_j = -\min_{v \in \Gamma_\infty(f)} \langle w_j, v \rangle \geq 0 \qquad (j = 1, 2, \dots, n) \tag{5.5}$$

and $f_{\sigma_0}(y)$ is a polynomial on $\mathbb{C}^n(\sigma_0)$. Set $d = \gcd(c_1, \dots, c_k) := \gcd(\{c_j \mid 1 \leq j \leq k,\ c_j \neq 0\}) > 0$. Then in $\mathbb{C}_t^* \times T_\sigma \simeq (\mathbb{C}^*)^{n-\dim\sigma+1}_{t, y_{k+1}, \dots, y_n}$ we have

$$\widetilde{T_\sigma^\circ} = \{(t, y_{k+1}, \dots, y_n) \mid t^{-d} y_{k+1}^{-c_{k+1}} \cdots y_n^{-c_n} \times (f_{\sigma_0}|_{T_\sigma})(y_{k+1}, \dots, y_n) = 1\} \tag{5.6}$$

and the action Ψ_σ of the generator of the cyclic group μ_d on it is given by the multiplication of $(\zeta_d, 1, \dots, 1) \in \mathbb{C}_t^* \times T_\sigma$. To show that the hypersurfaces $\widetilde{T_\sigma^\circ} \subset \mathbb{C}_t^* \times T_\sigma$ are non-degenerate, we use the following elementary lemma.

Lemma 5.2 *Let g_0 be a Laurent polynomial on $(\mathbb{C}^*)^n$ such that the hypersurface $Z^* = \{x \in (\mathbb{C}^*)^n \mid g_0(x) = 0\}$ is non-degenerate and x^v be a monomial. Then the set of complex numbers $\lambda \in \mathbb{C}$ such that the hypersurface $Z_\lambda^* = \{x \in (\mathbb{C}^*)^n \mid g_0(x) - \lambda x^v = 0\}$ is non-degenerate is open dense in $\mathbb{C}$.*

Proof. It suffices to apply the Bertini-Sard theorem to the map $\frac{g_0}{x^v}\colon (\mathbb{C}^*)^n \longrightarrow \mathbb{C}$. □

Proposition 5.3 *In the situation as above, the hypersurfaces $\widetilde{T_\sigma^\circ} \subset \mathbb{C}_t^* \times T_\sigma$ are non-degenerate.*

Proof. By the non-degeneracy at infinity of f and Lemma 5.2 there exists $\lambda \in \mathbb{C}^*$ such that the hypersurface

$$\{(t, y_{k+1}, \dots, y_n) \mid t^{-d} y_{k+1}^{-c_{k+1}} \cdots y_n^{-c_n} \times (f_{\sigma_0}|_{T_\sigma})(y_{k+1}, \dots, y_n) = \lambda\} \tag{5.7}$$

in $\mathbb{C}_t^* \times T_\sigma$ is non-degenerate. Since it is isomorphic to $\widetilde{T_\sigma^\circ}$ by the multiplication by $(\lambda', 1, \dots, 1) \in \mathbb{C}_t^* \times T_\sigma$ for λ' satisfying $(\lambda')^d = \lambda$, $\widetilde{T_\sigma^\circ}$ is also non-degenerate. □

Note that if $\dim\gamma(\sigma) = n - \dim\sigma$ the integer $d > 0$ above is equal to the lattice distance of $\gamma(\sigma)$ from the origin $0 \in \mathbb{R}^n$. Moreover in this case, by (5.6) the Newton polytope of the defining equation of $\widetilde{T_\sigma^\circ}$ in $\mathbb{C}_t^* \times T_\sigma \simeq (\mathbb{C}^*)^{n-\dim\sigma+1}$ is the convex hull of $\{0\} \sqcup \gamma(\sigma)$. For each face at infinity $\gamma \prec \Gamma_\infty(f)$ of $\Gamma_\infty(f)$, let $d_\gamma > 0$ be the lattice distance of γ from the origin $0 \in \mathbb{R}^n$ and Δ_γ the convex hull of $\{0\} \sqcup \gamma$ in $\mathbb{R}^n$. Let $\mathbb{L}(\Delta_\gamma)$ be the $(\dim\gamma + 1)$-dimensional linear subspace of $\mathbb{R}^n$ spanned by Δ_γ and consider the lattice $M_\gamma = \mathbb{Z}^n \cap \mathbb{L}(\Delta_\gamma) \simeq \mathbb{Z}^{\dim\gamma+1}$ in it. Then we set $T_{\Delta_\gamma} := \mathrm{Spec}(\mathbb{C}[M_\gamma]) \simeq (\mathbb{C}^*)^{\dim\gamma+1}$. Moreover let $\mathbb{L}(\gamma)$ be the smallest affine linear subspace of $\mathbb{R}^n$ containing γ and for $v \in M_\gamma$ define their lattice heights $\mathrm{ht}(v,\gamma) \in \mathbb{Z}$ from $\mathbb{L}(\gamma)$ in $\mathbb{L}(\Delta_\gamma)$ so that we have $\mathrm{ht}(0,\gamma) = d_\gamma > 0$. Then to the group homomorphism $M_\gamma \longrightarrow \mathbb{C}^*$ defined by $v \longmapsto \zeta_{d_\gamma}^{\mathrm{ht}(v,\gamma)}$ we can naturally associate an element $\tau_\gamma \in T_{\Delta_\gamma}$. We define a Laurent polynomial $g_\gamma = \sum_{v\in M_\gamma} b_v x^v$ on T_{Δ_γ} by

$$b_v = \begin{cases} a_v & (v \in \gamma), \\ -1 & (v = 0), \\ 0 & (\text{otherwise}), \end{cases} \tag{5.8}$$

where $f = \sum_{v\in\mathbb{Z}_+^n} a_v x^v$. Then we have $NP(g_\gamma) = \Delta_\gamma$, $\mathrm{supp} g_\gamma \subset \{0\} \sqcup \gamma$ and the hypersurface $Z_{\Delta_\gamma}^* = \{x \in T_{\Delta_\gamma} \mid g_\gamma(x) = 0\}$ is non-degenerate by the proof of Proposition 5.3. Since $Z_{\Delta_\gamma}^* \subset T_{\Delta_\gamma}$ is invariant by the multiplication $l_{\tau_\gamma} \colon T_{\Delta_\gamma} \xrightarrow{\sim} T_{\Delta_\gamma}$ by τ_γ, $Z_{\Delta_\gamma}^*$ admits an action of μ_{d_γ}. We thus obtain an element $[Z_{\Delta_\gamma}^*]$ of $\mathcal{M}_{\mathbb{C}}^{\hat{\mu}}$. By the construction of $[Z_{\Delta_\gamma}^*]$ the following lemma is obvious.

Lemma 5.4 *Let $\gamma \prec \Gamma_\infty(f)$ be a face at infinity of $\Gamma_\infty(f)$ and $\sigma \in \Sigma$ a cone whose supporting face $\gamma(\sigma)$ in $\Gamma_\infty(f)$ is γ. Assume that $\dim\gamma = n - \dim\sigma$. Then in the Grothendieck ring $\mathcal{M}_{\mathbb{C}}^{\hat{\mu}}$ we have the equality*

$$[\widetilde{T_\sigma^\circ}] = [Z_{\Delta_\gamma}^*]. \tag{5.9}$$

To rewrite Theorem 4.7 in terms of $\Gamma_\infty(f)$ we need the following result.

Proposition 5.5 *Let $\gamma \prec \Gamma_\infty(f)$ be a face at infinity of $\Gamma_\infty(f)$ and $\sigma_1, \sigma_2 \in \Sigma$ cones whose supporting faces $\gamma(\sigma_1)$ and $\gamma(\sigma_2)$ in $\Gamma_\infty(f)$ are the same and equal to γ. Then in the Grothendieck group $\mathrm{K}_0(\mathrm{HS}^{\mathrm{mon}})$ we have*

$$\chi_h((\mathbb{L}-1)^{\dim\sigma_1 - 1} \cdot [\widetilde{T_{\sigma_1}^\circ}]) = \chi_h((\mathbb{L}-1)^{\dim\sigma_2 - 1} \cdot [\widetilde{T_{\sigma_2}^\circ}]). \tag{5.10}$$

Proof. Without loss of generality we may assume that $\dim\sigma_1 \leq \dim\sigma_2$. Set $\dim\sigma_i = k_i$ $(i = 1, 2)$.
(**Step 1**): First we prove (5.10) in the case $\sigma_1 \prec \sigma_2$. Let $\sigma_0 \in \Sigma$ be an n-dimensional cone such that $\sigma_1 \prec \sigma_2 \prec \sigma_0$ and $\{w_1, w_2, \ldots, w_n\} \subset \mathbb{Z}^n$ the set of the primitive vectors on the edges of σ_0. We may assume that $w_1, w_2, \ldots, w_{k_i}$ generate σ_i for $i = 1, 2$. Then in the affine open subset $\mathbb{C}^n(\sigma_0) \simeq \mathbb{C}_y^n$ of X_Σ associated to σ_0 we have

$$T_{\sigma_i} = \{y \in \mathbb{C}^n(\sigma_0) \mid y_1 = \cdots = y_{k_i} = 0,\ y_{k_i+1}, \ldots, y_n \neq 0\} \tag{5.11}$$

for $i = 1, 2$. Moreover on $\mathbb{C}^n(\sigma_0) \simeq \mathbb{C}_y^n$ the function $\widetilde{g} = \frac{1}{f}$ has the form:

$$\widetilde{g}(y) = y_1^{c_1} \cdots y_n^{c_n} \times g_{\sigma_0}(y), \tag{5.12}$$

where we set

$$c_j = - \min_{v \in \Gamma_\infty(f)} \langle w_j, v \rangle \geq 0 \qquad (j = 1, 2, \dots, n) \tag{5.13}$$

and $g_{\sigma_0}(y)$ is a meromorphic function on $\mathbb{C}^n(\sigma_0)$. By the assumption $\gamma(\sigma_1) = \gamma(\sigma_2) = \gamma$, the restriction $g_{\sigma_0}|_{T_{\sigma_1}}$ of g_{σ_0} to the larger torus T_{σ_1} depends only on the variables $y_{k_2+1}, \dots, y_n$. Set $d_i = \gcd(c_1, \dots, c_{k_i}) := \gcd(\{c_j \mid 1 \leq j \leq k_i,\ c_j \neq 0\}) > 0$ $(i = 1, 2)$. Then we have

$$\widetilde{T_{\sigma_i}^{\circ}} = \{(t_i, y_{k_i+1}, \dots, y_n) \mid t_i^{d_i} \cdot y_{k_i+1}^{c_{k_i+1}} \cdots y_n^{c_n} \times (g_{\sigma_0}|_{T_{\sigma_i}})(y_{k_2+1}, \dots, y_n) = 1\} \tag{5.14}$$

in $\mathbb{C}^*_{t_i} \times T_{\sigma_i} \simeq (\mathbb{C}^*)^{n-k_i+1}_{t_i, y_{k_i+1}, \dots, y_n}$ for $i = 1, 2$. By the relation $d_2 = \gcd(d_1, c_{k_1+1}, \dots, c_{k_2})$, for the integer $d = \frac{d_1}{d_2} \in \mathbb{Z}$ we have $\gcd(d, \frac{c_{k_1+1}}{d_2}, \dots, \frac{c_{k_2}}{d_2}) = 1$. Now let $A \in \{B \in M_{k_2-k_1+1}(\mathbb{Z}) \mid \det B = 1\}$ be a unimodular matrix whose first row is the primitive vector $(d, \frac{c_{k_1+1}}{d_2}, \dots, \frac{c_{k_2}}{d_2}) \in \mathbb{Z}^{k_2-k_1+1}$. Consider the automorphism Λ_A of the algebraic torus $\mathbb{C}^* \times (\mathbb{C}^*)^{k_2-k_1} \simeq (\mathbb{C}^*)^{k_2-k_1+1}$ defined by A:

$$(t_1, y_{k_1+1}, \dots, y_{k_2}) \longmapsto (t_2, z_{k_1+1}, \dots, z_{k_2}). \tag{5.15}$$

By this construction of Λ_A obviously we have $t_2 = t_1^d y_{k_1+1}^{\frac{c_{k_1+1}}{d_2}} \cdots y_{k_2}^{\frac{c_{k_2}}{d_2}}$. Therefore the automorphism $\Lambda_A \times \mathrm{id}_{T_{\sigma_2}}$ of $\mathbb{C}^* \times (\mathbb{C}^*)^{k_2-k_1} \times T_{\sigma_2}$ induces an isomorphism

$$\widetilde{T_{\sigma_1}^{\circ}} \xrightarrow{\sim} (\mathbb{C}^*)^{k_2-k_1}_{z_{k_1+1}, \dots, z_{k_2}} \times \widetilde{T_{\sigma_2}^{\circ}}. \tag{5.16}$$

Moreover we have $\Lambda_A(\zeta_{d_1}, 1, \dots, 1) = (\zeta_{d_2}, \beta_{k_1+1}, \dots, \beta_{k_2})$ for some $\beta_i \in \mathbb{C}^*$. Since the action Ψ_{σ_1} of the generator of μ_{d_1} on $\widetilde{T_{\sigma_1}^{\circ}}$ is the multiplication by the element $(\zeta_{d_1}, 1, \dots, 1) \in \mathbb{C}^* \times T_{\sigma_1}$, the automorphism of $(\mathbb{C}^*)^{k_2-k_1} \times \widetilde{T_{\sigma_2}^{\circ}}$ induced by Ψ_{σ_1} via (5.16) is given by

$$(z_{k_1+1}, \dots, z_{k_2}, t_2, y_{k_2+1}, \dots, y_n) \longmapsto (\beta_{k_1+1} z_{k_1+1}, \dots, \beta_{k_2} z_{k_2}, \zeta_{d_2} t_2, y_{k_2+1}, \dots, y_n). \tag{5.17}$$

This is obviously homotopic to $\mathrm{id}_{(\mathbb{C}^*)^{k_2-k_1}} \times \Psi_{\sigma_2}$. Hence in the Grothendieck group $\mathrm{K}_0(\mathrm{HS}^{\mathrm{mon}})$ we obtain the equality

$$\chi_h([\widetilde{T_{\sigma_1}^{\circ}}]) = \chi_h((\mathbb{L} - 1)^{k_2-k_1} \cdot [\widetilde{T_{\sigma_2}^{\circ}}]), \tag{5.18}$$

from which (5.10) follows immediately.
(**Step 2**): Finally let us prove (5.10) in the general case. Let σ be the unique cone in the dual fan Σ_1 of $\Gamma_\infty(f)$ whose supporting face in $\Gamma_\infty(f)$ is γ. Then our assumption $\gamma(\sigma_1) = \gamma(\sigma_2) = \gamma$ implies that $\mathrm{rel.int}(\sigma_i) \subset \mathrm{rel.int}(\sigma)$ for $i = 1, 2$. So there exists a continuous curve in $\mathrm{rel.int}(\sigma)$ which starts from a point in $\mathrm{rel.int}(\sigma_1)$ and ends at the one in $\mathrm{rel.int}(\sigma_2)$. Then applying (**Step 1**) to each pair of two adjacent cones on it, we obtain (5.10). This completes the proof. □

Remark 5.6 In Proposition 5.5 if $\dim\sigma_1 \leq \dim\sigma_2$ we can prove also a slightly stronger equality

$$\chi_h([\widetilde{T_{\sigma_1}^{\circ}}]) = \chi_h((\mathbb{L} - 1)^{\dim\sigma_2 - \dim\sigma_1} \cdot [\widetilde{T_{\sigma_2}^{\circ}}]). \tag{5.19}$$

Since we do not use it in this paper, we omit the proof.

For a face at infinity $\gamma \prec \Gamma_\infty(f)$ let $S_\gamma \subset \{1, 2, \dots, n\}$ be the minimal subset of $\{1, 2, \dots, n\}$ such that $\gamma \subset \mathbb{R}^{S_\gamma}$ and set $m_\gamma = \sharp S_\gamma - \dim \gamma - 1 \geq 0$.

Theorem 5.7 *In the situation as above, we have the following results, where in the sums* $\sum_\gamma$ *below the face* γ *of* $\Gamma_\infty(f)$ *ranges through those at infinity.*

(i) *In the Grothendieck group* $\mathrm{K}_0(\mathrm{HS}^{\mathrm{mon}})$*, we have*

$$[H_f^\infty] = \chi_h(\mathcal{S}_f^\infty) = \sum_\gamma \chi_h((1-\mathbb{L})^{m_\gamma} \cdot [Z^*_{\Delta_\gamma}]). \tag{5.20}$$

(ii) *Let* $\lambda \in \mathbb{C}^* \setminus \{1\}$ *and* $k \geq 1$*. Then the number of the Jordan blocks for the eigenvalue* λ *with sizes* $\geq k$ *in* $\Phi_{n-1}^\infty \colon H^{n-1}(f^{-1}(R);\mathbb{C}) \xrightarrow{\sim} H^{n-1}(f^{-1}(R);\mathbb{C})$ $(R \gg 0)$ *is equal to*

$$(-1)^{n-1} \sum_{p+q=n-2+k, n-1+k} \left\{ \sum_\gamma e^{p,q} \left(\chi_h((1-\mathbb{L})^{m_\gamma} \cdot [Z^*_{\Delta_\gamma}]) \right)_\lambda \right\}. \tag{5.21}$$

(iii) *For* $k \geq 1$*, the number of the Jordan blocks for the eigenvalue* 1 *with sizes* $\geq k$ *in* Φ_{n-1}^∞ *is equal to*

$$(-1)^{n-1} \sum_{p+q=n-2-k, n-1-k} \left\{ \sum_\gamma e^{p,q} \left(\chi_h((1-\mathbb{L})^{m_\gamma} \cdot [Z^*_{\Delta_\gamma}]) \right)_1 \right\}. \tag{5.22}$$

Proof. (i) It suffices to rewrite Theorem 4.7. Let γ be a face at infinity of $\Gamma_\infty(f)$ such that $\sharp S_\gamma = n$. Denote by σ the unique $(n - \dim\gamma)$-dimensional cone in the dual fan Σ_1 of $\Gamma_\infty(f)$ whose supporting face in $\Gamma_\infty(f)$ is γ. Let Σ be the smooth subdivision of Σ_1 in the proof of Theorem 3.6 and σ_j $(1 \leq j \leq l)$ the cones in Σ such that $\mathrm{rel.int}(\sigma_j) \subset \mathrm{rel.int}(\sigma)$. Recall that T_{σ_j} is the $(n - \dim\sigma_j)$-dimensional T-orbit in X_Σ which corresponds to $\sigma_j \in \Sigma$ and we set $T^\circ_{\sigma_j} = T_{\sigma_j} \setminus \overline{f^{-1}(0)}$. Then in the motivic Milnor fiber at infinity $\mathcal{S}_f^\infty \in \mathcal{M}_{\mathbb{C}}^{\hat{\mu}}$ of f constructed by using the toric compactification X_Σ of $\mathbb{C}^n$, the following elements of $\mathcal{M}_{\mathbb{C}}^{\hat{\mu}}$

$$(1-\mathbb{L})^{\dim \sigma_j - 1} \cdot [\widetilde{T^\circ_{\sigma_j}}] \in \mathcal{M}_{\mathbb{C}}^{\hat{\mu}} \qquad (1 \leq j \leq l) \tag{5.23}$$

are contained, where $\widetilde{T^\circ_{\sigma_j}}$ is the unramified Galois covering of $T^\circ_{\sigma_j}$. Let us fix $1 \leq j_0 \leq l$ such that $\dim\sigma_{j_0} = \dim\sigma = n - \dim\gamma$. Then by Proposition 5.5 for any $1 \leq j \leq l$ we have the equality

$$\chi_h((1-\mathbb{L})^{\dim \sigma_j - 1} \cdot [\widetilde{T^\circ_{\sigma_j}}]) = (-1)^{\dim\sigma_j - \dim\sigma} \cdot \chi_h((1-\mathbb{L})^{n-\dim\gamma-1} \cdot [\widetilde{T^\circ_{\sigma_{j_0}}}]) \tag{5.24}$$

in the Grothendieck group $\mathrm{K}_0(\mathrm{HS}^{\mathrm{mon}})$. Combining (5.24) with the obvious combinatorial identity $\sum_{j=1}^l (-1)^{\dim\sigma_j - \dim\sigma} = 1$, we obtain a very simple formula

$$\sum_{j=1}^l \chi_h((1-\mathbb{L})^{\dim \sigma_j - 1} \cdot [\widetilde{T^\circ_{\sigma_j}}]) = \chi_h((1-\mathbb{L})^{n-\dim\gamma-1} \cdot [\widetilde{T^\circ_{\sigma_{j_0}}}]). \tag{5.25}$$

Hence by Lemma 5.4, for the face at infinity γ of $\Gamma_\infty(f)$ such that $\sharp S_\gamma = n$ the equality

$$\sum_{j=1}^l \chi_h((1-\mathbb{L})^{\dim \sigma_j - 1} \cdot [\widetilde{T^\circ_{\sigma_j}}]) = \chi_h((1-\mathbb{L})^{m_\gamma} \cdot [Z^*_{\Delta_\gamma}]) \tag{5.26}$$

holds. In the same way, we can show similar equalities also for the faces at infinity γ of $\Gamma_\infty(f)$ such that $\sharp S_\gamma < n$. More precisely, for such γ let $(\mathbb{R}^{S_\gamma})^\perp \simeq \mathbb{R}^{n-\sharp S_\gamma}$ be the orthogonal complement of $\mathbb{R}^{S_\gamma} \subset \mathbb{R}^n$ in $(\mathbb{R}^n)^*$. Then some σ_j in the $(n-\dim\gamma)$-dimensional cone σ associated with γ may not satisfy the condition $(\mathbb{R}^{S_\gamma})^\perp \prec \sigma_j$. We can prove a formula similar to (5.26) by dividing the set of the cones σ_j into $\{\sigma_j \mid (\mathbb{R}^{S_\gamma})^\perp \prec \sigma_j\}$ and $\{\sigma_j \mid (\mathbb{R}^{S_\gamma})^\perp \not\prec \sigma_j\}$. We omit the detail. This completes the proof of the assertion (i). The assertions (ii) and (iii) can be deduced from (i) and Theorem 4.6. □

Remark 5.8 Since we used a homotopy in proving (5.24), we can prove the equality of Theorem 5.7 (i) only in the Grothendieck group $\mathrm{K}_0(\mathrm{HS}^{\mathrm{mon}})$ of Hodge structures. See also Remark 4.10.

Note that by using the results in Section 2 we can always calculate $e^{p,q}(\chi_h((1-\mathbb{L})^{m_\gamma}\cdot[Z^*_{\Delta_\gamma}]))_\lambda$ explicitly. Here we shall give some closed formulas for the numbers of the Jordan blocks with large sizes in Φ^∞_{n-1}. First let us consider the numbers of the Jordan blocks for the eigenvalues $\lambda \in \mathbb{C}\setminus\{1\}$. Let $q_1,\dots,q_l$ (resp. $\gamma_1,\dots,\gamma_{l'}$) be the 0-dimensional (resp. 1-dimensional) faces of $\Gamma_\infty(f)$ such that $q_i \in \mathrm{Int}(\mathbb{R}^n_+)$ (resp. the relative interior rel.int(γ_i) of γ_i is contained in $\mathrm{Int}(\mathbb{R}^n_+)$). Obviously these faces are at infinity. For each q_i (resp. γ_i), denote by $d_i>0$ (resp. $e_i>0$) the lattice distance $\mathrm{dist}(q_i,0)$ (resp. $\mathrm{dist}(\gamma_i,0)$) of it from the origin $0\in\mathbb{R}^n$. For $1\le i\le l'$, let Δ_i be the convex hull of $\{0\}\sqcup\gamma_i$ in $\mathbb{R}^n$. Then for $\lambda\in\mathbb{C}\setminus\{1\}$ and $1\le i\le l'$ such that $\lambda^{e_i}=1$ we set

$$n(\lambda)_i = \sharp\{v\in\mathbb{Z}^n\cap\mathrm{rel.int}(\Delta_i) \mid \mathrm{ht}(v,\gamma_i)=k\} + \sharp\{v\in\mathbb{Z}^n\cap\mathrm{rel.int}(\Delta_i) \mid \mathrm{ht}(v,\gamma_i)=e_i-k\}, \tag{5.27}$$

where k is the minimal positive integer satisfying $\lambda=\zeta_{e_i}^k$ and for $v\in\mathbb{Z}^n\cap\mathrm{rel.int}(\Delta_i)$ we denote by $\mathrm{ht}(v,\gamma_i)$ the lattice height of v from the base γ_i of Δ_i.

Theorem 5.9 *Let f be as above and $\lambda\in\mathbb{C}^*\setminus\{1\}$. Then we have*

(i) *The number of the Jordan blocks for the eigenvalue λ with the maximal possible size n in $\Phi^\infty_{n-1}\colon H^{n-1}(f^{-1}(R);\mathbb{C}) \xrightarrow{\sim} H^{n-1}(f^{-1}(R);\mathbb{C})$ $(R\gg 0)$ is equal to $\sharp\{q_i \mid \lambda^{d_i}=1\}$.*

(ii) *The number of the Jordan blocks for the eigenvalue λ with size $n-1$ in Φ^∞_{n-1} is equal to $\sum_{i\colon\lambda^{e_i}=1} n(\lambda)_i$.*

Proof. (i) By Theorem 5.7 (ii), the number of the Jordan blocks for the eigenvalue $\lambda\in\mathbb{C}^*\setminus\{1\}$ with the maximal possible size n in Φ^∞_{n-1} is

$$(-1)^{n-1}e^{n-1,n-1}(\chi_h(\mathcal{S}^\infty_f))_\lambda = (-1)^{n-1}\sum_{i=1}^{l} e^{n-1,n-1}(\chi_h((1-\mathbb{L})^{n-1}\cdot[Z^*_{\Delta_{q_i}}]))_\lambda \tag{5.28}$$

$$= \sum_{i=1}^{l} e^{0,0}(\chi_h([Z^*_{\Delta_{q_i}}]))_\lambda. \tag{5.29}$$

Note that $Z^*_{\Delta_{q_i}}$ is a finite subset of $\mathbb{C}^*$ consisting of d_i points. Then (i) follows from

$$\sum_{i=1}^{l} e^{0,0}(\chi_h([Z^*_{\Delta_{q_i}}]))_\lambda = \sharp\{q_i \mid \lambda^{d_i}=1\}. \tag{5.30}$$

The assertion (ii) can be proved similarly by expressing $e^{n-1,n-2}(\chi_h(\mathcal{S}^\infty_f))_\lambda + e^{n-2,n-1}(\chi_h(\mathcal{S}^\infty_f))_\lambda$ in terms of the 1-dimensional faces at infinity γ_i of $\Gamma_\infty(f)$. □

Example 5.10 Let $f(x,y) \in \mathbb{C}[x,y]$ be a convenient polynomial whose Newton polyhedron at infinity $\Gamma_\infty(f)$ has the following shape.

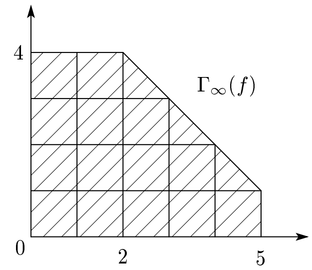

Figure 13

Assume moreover that f is non-degenerate at infinity. Then by Libgober-Sperber's theorem (Theorem 3.6) the characteristic polynomial $P(\lambda)$ of $\Phi_1^\infty \colon H^1(f^{-1}(R);\mathbb{C}) \xrightarrow{\sim} H^1(f^{-1}(R);\mathbb{C})$ $(R \gg 0)$ is given by

$$P(\lambda) = (\lambda - 1)(\lambda^4 - 1)(\lambda^6 - 1)^3. \tag{5.31}$$

This implies that the multiplicities of the roots of $P(\lambda) = 0$ are as follows:

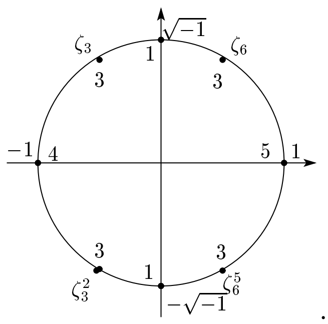

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Figure 14

For $\lambda \in \mathbb{C}$, denote by $H^1(f^{-1}(R);\mathbb{C})_\lambda$ the generalized λ-eigenspace of the monodromy operator Φ_1^∞ at infinity. First, by the monodromy theorem the restriction of Φ_1^∞ to $H^1(f^{-1}(R);\mathbb{C})_1 \simeq \mathbb{C}^5$ is semisimple. Moreover by Theorem 5.9 (i) the Jordan normal form of the restriction of Φ_1^∞ to $H^1(f^{-1}(R);\mathbb{C})_{-1} \simeq \mathbb{C}^4$ is

$$\begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{5.32}$$

In the same way, we can show that for $\lambda = \zeta_6, \sqrt{-1}, \zeta_3, \zeta_3^2, -\sqrt{-1}, \zeta_6^5$ the restriction of Φ_1^∞ to $H^1(f^{-1}(R);\mathbb{C})_\lambda$ is semisimple.

Next we consider the number of the Jordan blocks for the eigenvalue 1 in Φ_{n-1}^∞. By Proposition 2.9, we can rewrite Theorem 5.7 (iii) as follows. Denote by Π_f the number of the lattice points on the 1-skeleton of $\partial\Gamma_\infty(f) \cap \mathrm{Int}(\mathbb{R}_+^n)$.

Theorem 5.11 *In the situation as above, the number of the Jordan blocks for the eigenvalue* 1 *with the maximal possible size* $n-1$ *in* Φ_{n-1}^{∞} *is* Π_f.

Proof. For a face at infinity $\gamma \prec \Gamma_\infty(f)$, denote by $\Pi(\gamma)$ the number of the lattice points on the 1-skeleton of γ. Since for each face at infinity $\gamma \prec \Gamma_\infty(f)$ we have $\Pi(\Delta_\gamma)_1 - 1 = \Pi(\gamma)$ (for the definition of $\Pi(\Delta_\gamma)_1$, see Section 2), the assertion follows from Theorem 5.7 (iii) and Proposition 2.9. □

For a face at infinity $\gamma \prec \Gamma_\infty(f)$, denote by $l^*(\gamma)$ the number of the lattice points on the relative interior $\mathrm{rel.int}(\gamma)$ of γ. Then by Theorem 5.7 (iii) and Proposition 2.8, we also obtain the following result.

Theorem 5.12 *In the situation as above, the number of the Jordan blocks for the eigenvalue* 1 *with size* $n-2$ *in* Φ_{n-1}^{∞} *is equal to* $2\sum_\gamma l^*(\gamma)$, *where* γ *ranges through the faces at infinity of* $\Gamma_\infty(f)$ *such that* $\dim\gamma = 2$ *and* $\mathrm{rel.int}(\gamma) \subset \mathrm{Int}(\mathbb{R}^n_+)$. *In particular, this number is even.*

From now on, we assume that any face at infinity $\gamma \prec \Gamma_\infty(f)$ is prime in the sense of Definition 2.10 (i) and rewrite Theorem 5.7 (ii) and (iii) more explicitly. First, recall that by Proposition 2.6 for $\lambda \in \mathbb{C}^* \setminus \{1\}$ and a face at infinity $\gamma \prec \Gamma_\infty(f)$ we have $e^{p,q}(Z^*_{\Delta_\gamma})_\lambda = 0$ for any $p, q \geq 0$ such that $p+q > \dim\Delta_\gamma - 1 = \dim\gamma$. So the non-negative integers $r \geq 0$ such that $\sum_{p+q=r} e^{p,q}(Z^*_{\Delta_\gamma})_\lambda \neq 0$ are contained in the closed interval $[0, \dim\gamma] \subset \mathbb{R}$.

Definition 5.13 For a face at infinity $\gamma \prec \Gamma_\infty(f)$ and $k \geq 1$, we define a finite subset $J_{\gamma,k} \subset [0, \dim\gamma] \cap \mathbb{Z}$ by

$$J_{\gamma,k} = \{0 \leq r \leq \dim\gamma \mid n-2+k \equiv r \mod 2\}. \tag{5.33}$$

For each $r \in J_{\gamma,k}$, set

$$d_{k,r} = \frac{n-2+k-r}{2} \in \mathbb{Z}_+. \tag{5.34}$$

Since for any face at infinity $\gamma \prec \Gamma_\infty(f)$ the polytope Δ_γ is pseudo-prime in the sense of Definition 2.10 (ii), by Corollary 2.15 for $\lambda \in \mathbb{C}^* \setminus \{1\}$ and an integer $r \geq 0$ such that $r \in [0, \dim\gamma]$ we have

$$\sum_{p+q=r} e^{p,q}(\chi_h([Z^*_{\Delta_\gamma}]))_\lambda = (-1)^{\dim\gamma+r+1} \sum_{\substack{\Gamma \prec \Delta_\gamma \\ \dim\Gamma = r+1}} \left\{ \sum_{\Gamma' \prec \Gamma} (-1)^{\dim\Gamma'} \widetilde{\varphi}_\lambda(\Gamma') \right\}. \tag{5.35}$$

For simplicity, we denote this last integer by $e(\gamma, \lambda)_r$. Then by Theorem 5.7 (ii) we obtain the following result.

Theorem 5.14 *In the situation as above, let* $\lambda \in \mathbb{C}^* \setminus \{1\}$ *and* $k \geq 1$. *Then the number of the Jordan blocks for the eigenvalue* λ *with sizes* $\geq k$ *in* $\Phi_{n-1}^{\infty} \colon H^{n-1}(f^{-1}(R); \mathbb{C}) \xrightarrow{\sim} H^{n-1}(f^{-1}(R); \mathbb{C})$ $(R \gg 0)$ *is equal to*

$$(-1)^{n-1} \sum_\gamma \left\{ \sum_{r \in J_{\gamma,k}} (-1)^{d_{k,r}} \binom{m_\gamma}{d_{k,r}} \cdot e(\gamma, \lambda)_r + \sum_{r \in J_{\gamma,k+1}} (-1)^{d_{k+1,r}} \binom{m_\gamma}{d_{k+1,r}} \cdot e(\gamma, \lambda)_r \right\}, \tag{5.36}$$

where in the sum $\sum_\gamma$ the face γ of $\Gamma_\infty(f)$ ranges through those at infinity (we used also the convention $\binom{a}{b}=0$ $(0\leq a<b)$ for binomial coefficients).

By combining the proof of [5, Theorem 5.6] and Proposition 2.14 with Theorem 5.7 (iii), if any face at infinity $\gamma \prec \Gamma_\infty(f)$ is prime we can also explicitly describe the number of the Jordan blocks for the eigenvalue 1 in Φ_{n-1}^∞.

Finally to end this section, we prove a global analogue of the Steenbrink conjecture proved by Varchenko-Khovanskii [50] and Saito [41]. We return to the general case.

Definition 5.15 (Sabbah [36] and Steenbrink-Zucker [45]) As a Puiseux series, we define the spectrum at infinity $\mathrm{sp}_f^\infty(t)$ of f by

$$\mathrm{sp}_f^\infty(t)=\sum_{\beta\in(0,1]\cap\mathbb{Q}}\left[\sum_{i=0}^{n-1}(-1)^{n-1}\left\{\sum_{q\geq 0}e^{i,q}([H_f^\infty])_{\exp(2\pi\sqrt{-1}\beta)}\right\}t^{i+\beta}\right]+(-1)^nt^n. \tag{5.37}$$

When f is tame at infinity, by Theorem 4.5 we can easily prove that the support of $\mathrm{sp}_f^\infty(t)$ is contained in the open interval $(0,n)$ and has the symmetry

$$\mathrm{sp}_f^\infty(t)=t^n\mathrm{sp}_f^\infty\left(\frac{1}{t}\right) \tag{5.38}$$

with center at $\frac{n}{2}$. From now on, we assume that f is convenient and non-degenerate at infinity. In order to describe $\mathrm{sp}_f^\infty(t)$ by $\Gamma_\infty(f)$, for each face at infinity γ of $\Gamma_\infty(f)$ let $s_\gamma=\sharp S_\gamma\in\mathbb{Z}_{\geq 1}$ be the dimension of the minimal coordinate plane containing γ and set $\mathrm{Cone}(\gamma)=\mathbb{R}_+\gamma$. Next, let $h_f\colon \mathbb{R}_+^n\longrightarrow\mathbb{R}$ be the continuous function on $\mathbb{R}_+^n$ which is linear on each cone $\mathrm{Cone}(\gamma)$ and satisfies the condition $h_f|_{\partial\Gamma_\infty(f)\cap\mathrm{Int}(\mathbb{R}_+^n)}\equiv 1$. For a face at infinity γ of $\Gamma_\infty(f)$, let L_γ be the semigroup $\mathrm{Cone}(\gamma)\cap\mathbb{Z}_+^n$ and define its Poincaré series $P_\gamma(t)$ by

$$P_\gamma(t)=\sum_{\beta\in\mathbb{Q}_+}\sharp\{v\in L_\gamma \mid h_f(v)=\beta\}t^\beta. \tag{5.39}$$

Theorem 5.16 *In the situation as above, we have*

$$\mathrm{sp}_f^\infty(t)=\sum_\gamma(-1)^{n-1-\dim\gamma}(1-t)^{s_\gamma}P_\gamma(t)+(-1)^n, \tag{5.40}$$

where in the above sum γ ranges through the faces at infinity of $\Gamma_\infty(f)$.

Proof. For $\beta\in(0,1]\cap\mathbb{Q}$ and a face at infinity γ of $\Gamma_\infty(f)$, set

$$P_{\gamma,\beta}(t)=\begin{cases}\sum_{i=0}^\infty\sharp\{v\in L_\gamma\mid h_f(v)=i+\beta\}t^{i+\beta} & (0<\beta<1),\\ \sum_{i=0}^\infty\sharp\{v\in L_\gamma\mid h_f(v)=i\}t^i & (\beta=1)\end{cases} \tag{5.41}$$

so that we have

$$\sum_{\beta\in(0,1]\cap\mathbb{Q}}P_{\gamma,\beta}(t)=P_\gamma(t). \tag{5.42}$$

Then for $\beta \in \mathbb{Q}$ such that $0 < \beta < 1$ and a face at infinity γ of $\Gamma_\infty(f)$, by Theorem 2.7 and (2.20) we have

$$\begin{aligned}
&\sum_{i=0}^{n-1}(-1)^{n-1}\left\{\sum_{q\geq 0} e^{i,q}(\chi_h((1-\mathbb{L})^{m_\gamma}\cdot[Z^*_{\Delta_\gamma}]))_{\exp(2\pi\sqrt{-1}\beta)}\right\}t^{i+\beta}\\
&= (-1)^{n-1-\dim\gamma}t^\beta(1-t)^{m_\gamma}\sum_{i\geq 0}\varphi_{\exp(2\pi\sqrt{-1}\beta),\dim\gamma+1-i}(\Delta_\gamma)t^i \qquad (5.43)\\
&= (-1)^{n-1-\dim\gamma}t^\beta(1-t)^{s_\gamma-1-\dim\gamma}\frac{1}{t}\sum_{i\geq 0}\psi_{\exp(-2\pi\sqrt{-1}\beta),i+1}(\Delta_\gamma)t^{i+1} \qquad (5.44)\\
&= (-1)^{n-1-\dim\gamma}t^\beta(1-t)^{s_\gamma+1}\sum_{k\geq 1}l(k\Delta_\gamma)_{\exp(-2\pi\sqrt{-1}\beta)}t^{k-1} \qquad (5.45)\\
&= (-1)^{n-1-\dim\gamma}t^\beta(1-t)^{s_\gamma}\\
&\qquad \times(1-t)\left\{l(\Delta_\gamma)_{\exp(-2\pi\sqrt{-1}\beta)}+l(2\Delta_\gamma)_{\exp(-2\pi\sqrt{-1}\beta)}t+\cdots\right\} \qquad (5.46)\\
&= (-1)^{n-1-\dim\gamma}(1-t)^{s_\gamma}P_{\gamma,\beta}(t). \qquad (5.47)
\end{aligned}$$

Therefore, the assertion for the non-integral part of $\mathrm{sp}_f^\infty(t)$ follows immediately from Theorem 5.7 (i). Moreover, the integral part

$$\sum_{i=0}^{n-1}(-1)^{n-1}\left\{\sum_{q\geq 0} e^{i,q}(\chi_h(\mathcal{S}_f^\infty))_1\right\}t^{i+1}+(-1)^n t^n \qquad (5.48)$$

of sp_f^∞ is calculated as follows. For a face at infinity γ of $\Gamma_\infty(f)$, we have

$$\begin{aligned}
&\sum_{i=0}^{n-1}(-1)^{n-1}\left\{\sum_{q\geq 0} e^{i,q}(\chi_h((1-\mathbb{L})^{m_\gamma}\cdot[Z^*_{\Delta_\gamma}]))_1\right\}t^{i+1}\\
&= (-1)^{n-1-\dim\gamma}(1-t)^{m_\gamma}\sum_{i\geq 0}\left\{(-1)^i\binom{\dim\gamma+1}{i+1}+\varphi_{1,\dim\gamma+1-i}(\Delta_\gamma)\right\}t^{i+1} \qquad (5.49)\\
&= (-1)^{n-1-\dim\gamma}(1-t)^{s_\gamma-1-\dim\gamma}\left\{-(1-t)^{\dim\gamma+1}+1+\sum_{i\geq 0}\psi_{1,i+1}(\Delta_\gamma)t^{i+1}\right\} \qquad (5.50)\\
&= (-1)^{n-1-\dim\gamma}(1-t)^{s_\gamma-1-\dim\gamma}\\
&\qquad\times\Big[-(1-t)^{\dim\gamma+1}+(1-t)^{\dim\gamma+2}\left\{l(0)_1+l(\Delta_\gamma)_1t+l(2\Delta_\gamma)_1t^2+\cdots\right\}\Big] \qquad (5.51)\\
&= (-1)^{n-\dim\gamma}(1-t)^{s_\gamma}+(-1)^{n-1-\dim\gamma}(1-t)^{s_\gamma}P_{\gamma,1}(t). \qquad (5.52)
\end{aligned}$$

Summing up these terms over the faces at infinity γ of $\Gamma_\infty(f)$, we obtain

$$\sum_{i=0}^{n-1}(-1)^{n-1}\left\{\sum_{q\geq 0} e^{i,q}(\chi_h(\mathcal{S}_f^\infty))_1\right\}t^{i+1}=\sum_{\gamma}(-1)^{n-1-\dim\gamma}(1-t)^{s_\gamma}P_{\gamma,1}(t)+(-1)^{n+1}t^n+(-1)^n. \qquad (5.53)$$

This completes the proof. □

6 Sizes of Jordan blocks in monodromies at infinity

In this section, without assuming that f is tame at infinity, we prove some general results on the sizes and the numbers of the Jordan blocks in the monodromies at infinity Φ_j^∞ of

f. For this purpose, we inherit the situation and the notations at the beginning of Section 4. In this situation, the main result of Dimca-Saito [11] can be stated as follows.

Theorem 6.1 ([11, Theorem 0.1]) *Let $F_1, F_2, \dots, F_l$ be the irreducible components of $\widetilde{X} \setminus \mathbb{C}^n$ contained in $\widetilde{X} \setminus \Omega$. Assume that for generic complex numbers $c \in \mathbb{C}$ the closures $\overline{f^{-1}(c)}$ of $f^{-1}(c)$ in $\widetilde{X}$ are smooth and intersect $\bigcap_{i \in I} F_i$ for any $I \subset \{1, 2, \dots, l\}$ transversally. By taking such a complex number $c \in \mathbb{C}$ we set*

$$K = \max_{p \in (\widetilde{X} \setminus \Omega) \cap \overline{f^{-1}(c)}} \left(\sharp \{ F_i \mid p \in F_i \} \right). \tag{6.1}$$

Then the size of the Jordan blocks for the eigenvalue 1 of the monodromies at infinity $\Phi_j^\infty \colon H^j(f^{-1}(R); \mathbb{C}) \xrightarrow{\sim} H^j(f^{-1}(R); \mathbb{C})$ ($R \gg 0$, $j = 0, 1, \dots$) is bounded by K.

By using Saito's mixed Hodge modules in a different way, we can prove a similar result also for the eigenvalues $\lambda \in \mathbb{C} \setminus \{1\}$ of Φ_j^∞ as follows. Recall that the size of the Jordan blocks for such eigenvalues in Φ_j^∞ is bounded by $j+1$ by the monodromy theorem. Let $E_1, E_2, \dots, E_k$ be the irreducible components of the normal crossing divisor $U = \Omega \cap Y$ in $\Omega \subset \widetilde{X}$ and for $1 \leq i \leq k$ let $b_i > 0$ be the order of the poles of f along E_i as in Section 4. Moreover for $\lambda \in \mathbb{C}$ we set

$$R_\lambda = \{ 1 \leq i \leq k \mid \lambda^{b_i} = 1 \} \subset \{1, \dots, k\}. \tag{6.2}$$

Theorem 6.2 *Assume that $\lambda \in \mathbb{C} \setminus \{1\}$.*

(i) *We set*

$$K_\lambda = \max_{p \in U} \left(\sharp \{ E_i \mid p \in E_i \quad \text{and} \quad \lambda^{b_i} = 1 \} \right). \tag{6.3}$$

Then for any $0 \leq j \leq n-1$ the size of the Jordan blocks for the eigenvalue λ in Φ_j^∞ is bounded by K_λ.

(ii) *For $0 \leq j \leq n-1$, we set*

$$S(\lambda)_j = \{ I \subset R_\lambda \mid \sharp I = j+1 \quad \text{and} \quad E_I \neq \emptyset \}. \tag{6.4}$$

Moreover for each $I \in S(\lambda)_j$ denote by c_I the number of the connected components of E_I which do not intersect E_i for any $i \notin R_\lambda$. Then the number of the Jordan blocks for the eigenvalue λ with the maximal possible size $j+1$ in Φ_j^∞ is bounded by $\sum_{I \in S(\lambda)_j} c_I$.

Corollary 6.3 *We set*

$$S(\lambda) = \{ I \subset R_\lambda \mid \sharp I = n \quad \text{and} \quad E_I \neq \emptyset \} \tag{6.5}$$

and for each $I \in S(\lambda)$ denote by n_I the cardinality of the discrete (hence finite) set E_I. Then the number of the Jordan blocks for $\lambda \in \mathbb{C} \setminus \{1\}$ with the maximal possible size n in Φ_{n-1}^∞ is bounded by $\sum_{I \in S(\lambda)} n_I$.

Proof of Theorem 6.2 Recall the commutative diagram

$$\begin{tikzcd} \mathbb{C}^n \arrow[r, hook, "\iota"] \arrow[d, "f"'] & \widetilde{X} \arrow[d, "g"] \\ \mathbb{C} \arrow[r, hook, "j"] & \mathbb{P}^1 \end{tikzcd} \tag{6.6}$$

in which g is a proper holomorphic map. We set $\widetilde{g} = \frac{1}{f}$ as in Section 4. Then for $R \gg 0$ we can easily prove the isomorphisms

$$H^j(f^{-1}(R); \mathbb{C}) \simeq H^j(Y; \psi_{\widetilde{g}}(R\iota_* \mathbb{C}_{\mathbb{C}^n})) \simeq H^j(U; \psi_{\widetilde{g}}(\mathbb{C}_{\widetilde{X}})). \tag{6.7}$$

Now let us consider the perverse sheaf $\mathcal{F} = \psi_{\widetilde{g}}(\mathbb{C}_{\widetilde{X}}[n-1]) \in \mathbf{D}^b_c(Y)$ on the normal crossing divisor Y and its monodromy automorphism

$$\Phi := \Phi(\mathbb{C}_{\widetilde{X}}[n-1]) \colon \mathcal{F} \xrightarrow{\sim} \mathcal{F}. \tag{6.8}$$

Then for $R \gg 0$ we have a commutative diagram

$$\begin{tikzcd} H^j(f^{-1}(R); \mathbb{C}) \arrow[r, "\Phi_j^\infty"] \arrow[d, "\wr"] & H^j(f^{-1}(R); \mathbb{C}) \arrow[d, "\wr"] \\ H^{j-n+1}(U; \mathcal{F}) \arrow[r, "\Phi"] & H^{j-n+1}(U; \mathcal{F}). \end{tikzcd} \tag{6.9}$$

Moreover there exists a canonical decomposition

$$\mathcal{F} = \bigoplus_{\lambda \in \mathbb{C}} \mathcal{F}_\lambda, \tag{6.10}$$

where we set $\mathcal{F}_\lambda = \mathrm{Ker}\left[(\lambda \cdot \mathrm{id} - \Phi)^N \colon \mathcal{F} \longrightarrow \mathcal{F}\right]$ for $N \gg 0$. Therefore, for the given $\lambda \in \mathbb{C} \setminus \{1\}$ the generalized eigenspace for the eigenvalue λ in $\Phi_j^\infty \colon H^j(f^{-1}(R); \mathbb{C}) \xrightarrow{\sim} H^j(f^{-1}(R); \mathbb{C})$ $(R \gg 0)$ is isomorphic to $H^{j-n+1}(U; \mathcal{F}_\lambda)$. Now let $\Phi|_{\mathcal{F}_\lambda} = (\lambda \cdot \mathrm{id}) \times \Phi_u$ be the Jordan decomposition of $\Phi|_{\mathcal{F}_\lambda} \colon \mathcal{F}_\lambda \longrightarrow \mathcal{F}_\lambda$ (Φ_u is the unipotent part) and set

$$N_\lambda = \frac{1}{2\pi\sqrt{-1}} \log \Phi_u = \frac{1}{2\pi\sqrt{-1}} \sum_{i=1}^{N} \frac{(-1)^{i+1}}{i} (\Phi_u - \mathrm{id})^i \tag{6.11}$$

for $N \gg 0$. Then N_λ is a nilpotent endomorphism of the perverse sheaf $\mathcal{F}_\lambda$ and there exists an automorphism $M_\lambda \colon \mathcal{F}_\lambda \longrightarrow \mathcal{F}_\lambda$ such that

$$\Phi_u - \mathrm{id} = N_\lambda M_\lambda = M_\lambda N_\lambda. \tag{6.12}$$

This implies that if $(N_\lambda)^i = 0$ for some $i \geq 1$ then $(\lambda \cdot \mathrm{id} - \Phi|_{\mathcal{F}_\lambda})^i = \lambda^i (\mathrm{id} - \Phi_u)^i = 0$ and the size of the Jordan blocks for the eigenvalue λ in $\Phi_j^\infty \colon H^j(f^{-1}(R); \mathbb{C}) \xrightarrow{\sim} H^j(f^{-1}(R); \mathbb{C})$ is $\leq i$. Let W be the weight filtration of the mixed Hodge module associated with the perverse sheaf $\mathcal{F}_\lambda \oplus \mathcal{F}_{\overline{\lambda}}$ (see e.g. [12]). Then the assertion (i) follows from the primitive decomposition of the graded module $\mathrm{Gr}^W(\mathcal{F}_\lambda)$ given in [12, Section 1.4]. Finally let us prove (ii). By the above argument, the number of the Jordan blocks for the eigenvalue λ with the maximal possible size $j+1$ in Φ_j^∞ is equal to

$$\dim \left(\mathrm{Im} \left[H^{j-n+1}(U; \mathcal{F}_\lambda) \xrightarrow{N_\lambda^j} H^{j-n+1}(U; \mathcal{F}_\lambda) \right] \right). \tag{6.13}$$

Let $\mathcal{G}$ be the subobject $\mathrm{Im} N_\lambda^j$ of $\mathcal{F}_\lambda$ in the category of perverse sheaves on Y. Then we have a commutative diagram

$$\begin{tikzcd} H^{j-n+1}(U;\mathcal{F}_\lambda) \arrow[rr, "N_\lambda^j"] \arrow[dr] & & H^{j-n+1}(U;\mathcal{F}_\lambda) \\ & H^{j-n+1}(U;\mathcal{G}), \arrow[ur] & \end{tikzcd} \tag{6.14}$$

and the number (6.13) is bounded by $\dim H^{j-n+1}(U;\mathcal{G})$. Let us set $\mathcal{G}' = \mathcal{G} \cap W_{n-j-1}\mathcal{F}_\lambda$ and $\mathcal{G}'' = \mathcal{G} \cap W_{n-j-2}\mathcal{F}_\lambda$. Then by the exact sequence of perverse sheaves

$$0 \longrightarrow \mathcal{G}' \longrightarrow \mathcal{G} \longrightarrow \mathcal{G}/\mathcal{G}' \longrightarrow 0 \tag{6.15}$$

and $\dim \mathrm{supp}(\mathcal{G}/\mathcal{G}') \leq n-j-2$ we obtain an isomorphism

$$H^{j-n+1}(U;\mathcal{G}') \simeq H^{j-n+1}(U;\mathcal{G}). \tag{6.16}$$

Moreover it follows from the exact sequence of perverse sheaves

$$0 \longrightarrow \mathcal{G}'' \longrightarrow \mathcal{G}' \longrightarrow \mathcal{G}'/\mathcal{G}'' \simeq \mathrm{Gr}^W_{n-j-1}\mathcal{F}_\lambda \longrightarrow 0 \tag{6.17}$$

and $\dim \mathrm{supp}\mathcal{G}'' \leq n-j-2$ that we have

$$\dim H^{j-n+1}(U;\mathcal{G}') \leq \dim H^{j-n+1}(U;\mathrm{Gr}^W_{n-j-1}\mathcal{F}_\lambda). \tag{6.18}$$

Then we can use the primitive decomposition of $\mathrm{Gr}^W(\mathcal{F}_\lambda)$ in [12, Section 1.4] to estimate $\dim H^{j-n+1}(U;\mathrm{Gr}^W_{n-j-1}\mathcal{F}_\lambda)$. For $I \subset R_\lambda$ such that $\sharp I = j+1$, set $U_I = E_I \setminus \left(\bigcup_{i \notin R_\lambda} E_i\right)$. Then by the results in [12, Section 1.4] we can easily see that

$$\dim H^{j-n+1}(U;\mathrm{Gr}^W_{n-j-1}\mathcal{F}_\lambda) \leq \dim \left(\bigoplus_{\substack{I \subset R_\lambda \\ \sharp I = j+1}} \Gamma(U_I;\mathcal{L}_{\lambda,I}) \right), \tag{6.19}$$

where $\mathcal{L}_{\lambda,I}$ is a local system of rank one on U_I whose monodromy around the divisor E_i ($i \notin R_\lambda$) is given by the multiplication by $\lambda^{-b_i} (\neq 1)$ (see [12, Section 1.4] for the detail). If a connected component $E_{I,r}$ of E_I intersects E_i for some $i \notin R_\lambda$ we have $\Gamma(U_I \cap E_{I,r};\mathcal{L}_{\lambda,I}) = 0$. Therefore the assertion (ii) follows. This completes the proof. □

7 Some consequences in local Milnor monodromies

Our results in Section 2 and the arguments in Section 5 can be applied also to the nilpotent parts of local Milnor monodromies. Namely, we can rewrite Denef-Loeser's result [6, Theorem 4.2.1] as follows. Let $f \in \mathbb{C}[x_1, \ldots, x_n]$ be a polynomial such that the hypersurface $\{x \in \mathbb{C}^n \mid f(x) = 0\}$ has an isolated singular point at $0 \in \mathbb{C}^n$. Then by a fundamental theorem of Milnor [31], for the Milnor fiber F_0 of f at 0 we have $H^j(F_0;\mathbb{C}) \simeq 0$ ($j \neq 0, \ n-1$). In this situation, by using an embedded resolution of $\{x \in \mathbb{C}^n \mid f(x) = 0\}$, Denef-Loeser [6] and [7] introduced the motivic Milnor fiber $\mathcal{S}_{f,0} \in \mathcal{M}_{\mathbb{C}}^{\hat{\mu}}$ of f at $0 \in \mathbb{C}^n$ such that $\chi_h(\mathcal{S}_{f,0})$ coincides with the Hodge characteristic $[H_f] \in \mathrm{K}_0(\mathrm{HS}^{\mathrm{mon}})$ of F_0. For $[H_f] \in \mathrm{K}_0(\mathrm{HS}^{\mathrm{mon}})$ the following result due to Steenbrink [44] and Saito [40], [42] is fundamental.

Theorem 7.1 (Steenbrink [44] and Saito [40], [42]) *In the situation as above, we have*

(i) *Let $\lambda \in \mathbb{C}^* \setminus \{1\}$. Then we have $e^{p,q}([H_f])_\lambda = 0$ for $(p,q) \notin [0,n-1] \times [0,n-1]$. Moreover for $(p,q) \in [0,n-1] \times [0,n-1]$ we have*

$$e^{p,q}([H_f])_\lambda = e^{n-1-q,n-1-p}([H_f])_\lambda. \tag{7.1}$$

(ii) *We have $e^{p,q}([H_f])_1 = 0$ for $(p,q) \notin \{(0,0)\} \sqcup ([1,n-1] \times [1,n-1])$ and $e^{0,0}([H_f])_1 = 1$. Moreover for $(p,q) \in [1,n-1] \times [1,n-1]$ we have*

$$e^{p,q}([H_f])_1 = e^{n-q,n-p}([H_f])_1. \tag{7.2}$$

We can check these symmetries of $e^{p,q}([H_f])_\lambda$ by calculating $\chi_h(\mathcal{S}_{f,0}) \in \mathrm{K}_0(\mathrm{HS}^{\mathrm{mon}})$ explicitly by our methods (see below) in many cases. Since the weights of $[H_f] \in \mathrm{K}_0(\mathrm{HS}^{\mathrm{mon}})$ are defined by the monodromy filtration, we have the following result.

Theorem 7.2 *In the situation as above, we have*

(i) *Let $\lambda \in \mathbb{C}^* \setminus \{1\}$ and $k \geq 1$. Then the number of the Jordan blocks for the eigenvalue λ with sizes $\geq k$ in $\Phi_{n-1,0} \colon H^{n-1}(F_0;\mathbb{C}) \xrightarrow{\sim} H^{n-1}(F_0;\mathbb{C})$ is equal to*

$$(-1)^{n-1} \sum_{p+q=n-2+k,n-1+k} e^{p,q}(\chi_h(\mathcal{S}_{f,0}))_\lambda. \tag{7.3}$$

(ii) *For $k \geq 1$, the number of the Jordan blocks for the eigenvalue 1 with sizes $\geq k$ in $\Phi_{n-1,0}$ is equal to*

$$(-1)^{n-1} \sum_{p+q=n-1+k,n+k} e^{p,q}(\chi_h(\mathcal{S}_{f,0}))_1. \tag{7.4}$$

In order to rewrite Theorem 7.2 more explicitly, we prepare some notations.

Definition 7.3 Let $f(x) \in \mathbb{C}[x_1, \dots, x_n]$ be a polynomial on $\mathbb{C}^n$.

(i) We call the convex hull of $\bigcup_{v \in \mathrm{supp} f} \{v + \mathbb{R}^n_+\}$ in $\mathbb{R}^n_+$ the (usual) Newton polyhedron of f and denote it by $\Gamma_+(f)$.

(ii) The union of the compact faces of $\Gamma_+(f)$ is called the Newton boundary of f and denoted by Γ_f.

Definition 7.4 ([21]) We say that a polynomial $f(x) = \sum_{v \in \mathbb{Z}^n_+} a_v x^v$ ($a_v \in \mathbb{C}$) is non-degenerate at $0 \in \mathbb{C}^n$ if for any face $\gamma \prec \Gamma_+(f)$ such that $\gamma \subset \Gamma_f$ the complex hypersurface $\{x \in (\mathbb{C}^*)^n \mid f_\gamma(x) = 0\}$ in $(\mathbb{C}^*)^n$ is smooth and reduced, where we set $f_\gamma(x) = \sum_{v \in \gamma \cap \mathbb{Z}^n_+} a_v x^v$.

Recall that generic polynomials having a fixed Newton polyhedron are non-degenerate at $0 \in \mathbb{C}^n$. From now on, we assume also that $f \in \mathbb{C}[x_1, \dots, x_n]$ is convenient and non-degenerate at $0 \in \mathbb{C}^n$. For each face $\gamma \prec \Gamma_+(f)$ such that $\gamma \subset \Gamma_f$, let Δ_γ be the convex hull of $\{0\} \sqcup \gamma$ in $\mathbb{R}^n$. Then we define $m_\gamma \geq 0$, $d_\gamma > 0$, $\mathbb{L}(\gamma)$, M_γ, $T_{\Delta_\gamma} = \mathrm{Spec}(\mathbb{C}[M_\gamma])$, $Z^*_{\Delta_\gamma} \subset T_{\Delta_\gamma}$ and $\mathrm{ht}(v, \gamma) \in \mathbb{Z}$ for $v \in M_\gamma$ as in Section 5. But this time we define $\tau_\gamma \in T_{\Delta_\gamma}$ to be the element which corresponds to the group homomorphism $M_\gamma \longrightarrow \mathbb{C}^*$ defined by $v \longmapsto \zeta_{d_\gamma}^{-\mathrm{ht}(v,\gamma)}$. Then $Z^*_{\Delta_\gamma} \subset T_{\Delta_\gamma}$ is invariant by the multiplication $l_{\tau_\gamma} \colon T_{\Delta_\gamma} \xrightarrow{\sim} T_{\Delta_\gamma}$ by τ_γ, and hence we obtain an element $[Z^*_{\Delta_\gamma}]$ of $\mathcal{M}^{\hat{\mu}}_{\mathbb{C}}$. Moreover let $\mathbb{L}(\gamma)' \simeq \mathbb{R}^{\dim\gamma}$ be a linear subspace of $\mathbb{R}^n$ such that $\mathbb{L}(\gamma) = \mathbb{L}(\gamma)' + w$ for some $w \in \mathbb{Z}^n$ and set $\gamma' = \gamma - w \subset \mathbb{L}(\gamma)'$. We define a Laurent polynomial $g'_\gamma = \sum_{v \in \mathbb{L}(\gamma)' \cap \mathbb{Z}^n} b'_v x^v$ on $T(\gamma) := \mathrm{Spec}(\mathbb{C}[\mathbb{L}(\gamma)' \cap \mathbb{Z}^n]) \simeq (\mathbb{C}^*)^{\dim\gamma}$ by

$$b'_v = \begin{cases} a_{v+w} & (v \in \gamma'), \\ 0 & (\text{otherwise}), \end{cases} \tag{7.5}$$

where $f = \sum_{v \in \mathbb{Z}^n_+} a_v x^v$. Then we have $NP(g'_\gamma) = \gamma'$ and the hypersurface $Z^*_\gamma = \{x \in T(\gamma) \mid g'_\gamma(x) = 0\}$ is non-degenerate. We define $[Z^*_\gamma] \in \mathcal{M}^{\hat{\mu}}_{\mathbb{C}}$ to be the class of the variety Z^*_γ with the trivial action of $\hat{\mu}$.

Theorem 7.5 *In the situation as above, we have*

(i) *In the Grothendieck group* $\mathrm{K}_0(\mathrm{HS}^{\mathrm{mon}})$, *we have*

$$\chi_h(\mathcal{S}_{f,0}) = \sum_{\gamma \subset \Gamma_f} \chi_h\big((1-\mathbb{L})^{m_\gamma} \cdot [Z^*_{\Delta_\gamma}]\big) + \sum_{\substack{\gamma \subset \Gamma_f \\ \dim\gamma \geq 1}} \chi_h\big((1-\mathbb{L})^{m_\gamma+1} \cdot [Z^*_\gamma]\big). \tag{7.6}$$

(ii) *Let* $\lambda \in \mathbb{C}^* \setminus \{1\}$ *and* $k \geq 1$. *Then the number of the Jordan blocks for the eigenvalue* λ *with sizes* $\geq k$ *in* $\Phi_{n-1,0} \colon H^{n-1}(F_0; \mathbb{C}) \xrightarrow{\sim} H^{n-1}(F_0; \mathbb{C})$ *is equal to*

$$(-1)^{n-1} \sum_{p+q=n-2+k, n-1+k} \left\{ \sum_{\gamma \subset \Gamma_f} e^{p,q} \left(\chi_h \left((1-\mathbb{L})^{m_\gamma} \cdot [Z^*_{\Delta_\gamma}] \right) \right)_\lambda \right\}. \tag{7.7}$$

(iii) *For* $k \geq 1$, *the number of the Jordan blocks for the eigenvalue* 1 *with sizes* $\geq k$ *in* $\Phi_{n-1,0}$ *is equal to*

$$\begin{aligned}(-1)^{n-1} \sum_{p+q=n-1+k, n+k} \Bigg\{ &\sum_{\gamma \subset \Gamma_f} e^{p,q}\big(\chi_h\big((1-\mathbb{L})^{m_\gamma} \cdot [Z^*_{\Delta_\gamma}]\big)\big)_1 \\ &+ \sum_{\substack{\gamma \subset \Gamma_f \\ \dim\gamma \geq 1}} e^{p,q}\big(\chi_h\big((1-\mathbb{L})^{m_\gamma+1} \cdot [Z^*_\gamma]\big)\big)_1 \Bigg\}.\end{aligned} \tag{7.8}$$

Proof. Since (ii) and (iii) follow from (i) and Theorem 7.2, it suffices to prove (i). The proof is very similar to the one in Varchenko [49]. Let Σ_1 be the dual fan of $\Gamma_+(f)$ in $\mathbb{R}^n_+$ and Σ its smooth subdivision. Denote by X_Σ the smooth toric variety associated to Σ. Since the union of the cones in Σ is $\mathbb{R}^n_+$, there exists a proper morphism $\pi \colon X_\Sigma \longrightarrow \mathbb{C}^n$. By the convenience of f, we can construct the smooth fan Σ without subdividing the cones

contained in $\partial\mathbb{R}^n_+$. Then π induces an isomorphism $X_\Sigma \setminus \pi^{-1}(0) \simeq \mathbb{C}^n \setminus \{0\}$. Moreover by the non-degeneracy at $0 \in \mathbb{C}^n$ of f, the proper transform Z of the hypersurface $\{x \in \mathbb{C}^n \mid f(x) = 0\}$ in X_Σ is smooth and intersects T-orbits in $\pi^{-1}(0)$ transversally. Let $D_1, \dots, D_m$ be the toric divisors in X_Σ. For a non-empty subset $I \subset \{1, 2, \dots, m\}$ we set $D_I = \bigcap_{i \in I} D_i$ and

$$D_I^\circ = D_I \setminus \left\{ \left(\bigcup_{i \notin I} D_i \right) \cup Z \right\} \subset X_\Sigma. \tag{7.9}$$

Moreover we set

$$Z_I^\circ = \left\{ D_I \setminus \left(\bigcup_{i \notin I} D_i \right) \right\} \cap Z \subset X_\Sigma \tag{7.10}$$

and denote by $[Z_I^\circ] \in \mathcal{M}_{\mathbb{C}}^{\hat{\mu}}$ the class of the variety Z_I° with the trivial action. Then, unlike the global object $\mathcal{S}_f^\infty$, Denef-Loeser's "local" motivic Milnor fiber $\mathcal{S}_{f,0}$ contains not only $(1-\mathbb{L})^{\sharp I - 1}[\widetilde{D_I^\circ}]$ but also $(1-\mathbb{L})^{\sharp I}[Z_I^\circ]$ (see [6] and [7] for the details). These new elements yield the second term in the right hand side of (7.6). Finally, in the Grothendieck group $\mathrm{K}_0(\mathrm{HS}^{\mathrm{mon}})$ we can rewrite $\chi_h(\mathcal{S}_{f,0})$ in terms of the dual fan Σ_1 (i.e. in terms of $\Gamma_+(f)$) by using the proof of Theorem 5.7 (i). This completes the proof. $\square$

Let $q_1, \dots, q_l$ (resp. $\gamma_1, \dots, \gamma_{l'}$) be the 0-dimensional (resp. 1-dimensional) faces of $\Gamma_+(f)$ such that $q_i \in \mathrm{Int}(\mathbb{R}^n_+)$ (resp. $\mathrm{rel.int}(\gamma_i) \subset \mathrm{Int}(\mathbb{R}^n_+)$). Then by defining as in Section 5 the numbers $d_i > 0$ $(1 \leq i \leq l)$, $e_i > 0$ $(1 \leq i \leq l')$ and $n(\lambda)_i \geq 0$ $(1 \leq i \leq l')$ for $\lambda \in \mathbb{C} \setminus \{1\}$, we can easily obtain the following results from Theorem 7.5 (ii).

Theorem 7.6 *In the situation as above, for $\lambda \in \mathbb{C}^* \setminus \{1\}$, we have*

(i) *The number of the Jordan blocks for the eigenvalue λ with the maximal possible size n in $\Phi_{n-1,0}$ is equal to $\sharp\{q_i \mid \lambda^{d_i} = 1\}$.*

(ii) *The number of the Jordan blocks for the eigenvalue λ with size $n-1$ in $\Phi_{n-1,0}$ is equal to $\sum_{i:\, \lambda^{e_i}=1} n(\lambda)_i$.*

Remark 7.7 Assuming that any face $\gamma \prec \Gamma_+(f)$ such that $\gamma \subset \Gamma_f$ is prime in the sense of Definition 2.10 (i), we can obtain also some explicit formulas for the numbers of the Jordan blocks for the eigenvalues $\lambda \neq 1$ with smaller sizes $k \geq 1$ in $\Phi_{n-1,0}$. Since they are completely parallel to Theorem 5.14, we omit them. The results that we obtain in this way are different from the previous ones due to Danilov [4] and Tanabe [48]. For example, in [4] and [48] they assume a stronger condition that the Newton polyhedron $\Gamma_+(f)$ itself is prime. We could weaken this condition, because our Proposition 2.13 and Corollary 2.15 are generalizations of the corresponding results in [5] to pseudo-prime polytopes.

We can also obtain the corresponding results for the eigenvalue 1 by rewriting Theorem 7.5 (iii) more simply as follows.

Theorem 7.8 *In the situation as above, for $k \geq 1$ the number of the Jordan blocks for the eigenvalue 1 with sizes $\geq k$ in $\Phi_{n-1,0}$ is equal to*

$$(-1)^{n-1} \sum_{p+q=n-2-k,\, n-1-k} \left\{ \sum_{\gamma \subset \Gamma_f} e^{p,q} \left(\chi_h \left((1-\mathbb{L})^{m_\gamma} \cdot [Z^*_{\Delta_\gamma}] \right) \right)_1 \right\}. \tag{7.11}$$

By Theorem 7.8, we immediately obtain the following corollary. Denote by Π'_f the number of the lattice points on the 1-skeleton of $\Gamma_f \cap \mathrm{Int}(\mathbb{R}^n_+)$. Also, for a compact face $\gamma \prec \Gamma_+(f)$ we denote by $l^*(\gamma)$ the number of the lattice points on rel.int(γ).

Corollary 7.9 *In the situation as above, we have*

(i) *(van Doorn-Steenbrink [13]) The number of the Jordan blocks for the eigenvalue* 1 *with the maximal possible size $n-1$ in $\Phi_{n-1,0}$ is Π'_f.*

(ii) *The number of the Jordan blocks for the eigenvalue* 1 *with size $n-2$ in $\Phi_{n-1,0}$ is equal to $2\sum_\gamma l^*(\gamma)$, where γ ranges through the faces of $\Gamma_+(f)$ such that* $\dim\gamma = 2$ *and* rel.int$(\gamma) \subset \Gamma_f \cap \mathrm{Int}(\mathbb{R}^n_+)$.

Note that Corollary 7.9 (i) was previously obtained in van Doorn-Steenbrink [13] by different methods. Theorem 7.8 asserts that after replacing the faces at infinity of $\Gamma_\infty(f)$ by those of $\Gamma_+(f)$ contained in Γ_f the combinatorial description of the local monodromy $\Phi_{n-1,0}$ is the same as that of the global one Φ^∞_{n-1}. Namely we find a beautiful symmetry between local and global. Theorem 7.8 can be deduced from the following more precise result.

Theorem 7.10 *In the situation as above, for any $0 \leq p, q \leq n-2$ we have*

$$\begin{aligned}&\sum_{\gamma\subset\Gamma_f} e^{p,q}\left(\chi_h\left((1-\mathbb{L})^{m_\gamma}[Z^*_{\Delta_\gamma}]\right)\right)_1 \\ &= \sum_{\gamma\subset\Gamma_f} e^{p+1,q+1}\left(\chi_h\left((1-\mathbb{L})^{m_\gamma}[Z^*_{\Delta_\gamma}]+(1-\mathbb{L})^{m_\gamma+1}[Z^*_\gamma]\right)\right)_1. \qquad (7.12)\end{aligned}$$

We can easily see that Theorem 7.10 follows from Proposition 7.11 below. For $[V] \in \mathrm{K}_0(\mathrm{HS}^{\mathrm{mon}})$, let $e([V])_1 = \sum_{p,q=0}^\infty e^{p,q}([V])_1 t_1^p t_2^q$ be the generating function of $e^{p,q}([V])_1$ as in [5].

Proposition 7.11 *We have*

$$\sum_{\gamma\subset\Gamma_f} e\left(\chi_h\left((1-\mathbb{L})^{m_\gamma+1}([Z^*_{\Delta_\gamma}]+[Z^*_\gamma])\right)\right)_1 = 1-(t_1t_2)^n. \qquad (7.13)$$

From now on, we shall prove Proposition 7.11. First, we apply Proposition 2.12 to the case where $\Delta = \Delta_\gamma$ for a face γ of $\Gamma_+(f)$ such that $\gamma \subset \Gamma_f$. Let γ' be a prime polytope in $\mathbb{R}^{\dim\gamma}$ which majorizes γ and consider the Minkowski sum $\gamma'' := \gamma + \gamma'$ (resp. $\square_{\gamma''} := \Delta_\gamma + \gamma'$) in $\mathbb{R}^{\dim\gamma}$ (resp. $\mathbb{R}^{\dim\gamma+1}$). Then $\square_{\gamma''}$ is a $(\dim\gamma+1)$-dimensional truncated pyramid whose top (resp. bottom) is γ' (resp. γ'') (see Figure 15 below). In particular, $\square_{\gamma''}$ is prime. Since the dual fan of γ'' coincides with that of γ', the prime polytope γ'' majorizes γ. Let $\Psi\colon \mathrm{som}(\gamma'') \longrightarrow \mathrm{som}(\gamma)$ be the morphism between the sets of the vertices of γ'' and γ. By extending Ψ to a morphism $\widetilde{\Psi}\colon \mathrm{som}(\square_{\gamma''}) \longrightarrow \mathrm{som}(\Delta_\gamma)$ as

$$\widetilde{\Psi}(w) = \begin{cases}\Psi(w) & (w \in \mathrm{som}(\gamma'')),\\ \{0\} & (w \in \mathrm{som}(\gamma')),\end{cases} \qquad (7.14)$$

we see that the prime polytope $\square_{\gamma''}$ majorizes Δ_γ.

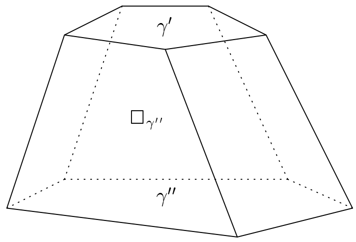

Figure 15

Proposition 7.12 *For the closure* $\overline{Z^*_{\Delta_\gamma}}$ *of* $Z^*_{\Delta_\gamma}$ *in* $X_{\square_{\gamma''}}$*, we have*

$$\sum_q e^{p,q}(\overline{Z^*_{\Delta_\gamma}})_1 = \sum_{\tau \prec \gamma''} (-1)^{\dim\tau + p} \binom{\dim\tau}{p}. \tag{7.15}$$

Proof. It suffices to rewrite Proposition 2.12 in this case. For a face Γ of $\square_{\gamma''}$, we set $b_\Gamma = \dim\Gamma - \dim\widetilde{\Psi}(\Gamma)$. Note that the set of faces of $\square_{\gamma''}$ consists of those of γ' and γ'' and side faces. Each side face of $\square_{\gamma''}$ is a truncated pyramid $\square_\tau$ whose bottom is $\tau \prec \gamma''$. Since $\dim\square_\tau = \dim\tau + 1$ and $b_{\square_\tau} = b_\tau$ for $\tau \prec \gamma''$, we have

$$\sum_{\Gamma \prec \square_{\gamma''}} (-1)^{\dim\Gamma + p + 1} \left\{ \binom{\dim\Gamma}{p+1} - \binom{b_\Gamma}{p+1} \right\} = \sum_{\tau \prec \gamma''} (-1)^{\dim\tau + p} \binom{\dim\tau}{p} \tag{7.16}$$

and

$$\begin{aligned} &\sum_{\Gamma \prec \square_{\gamma''}} (-1)^{\dim\Gamma + 1} \sum_{i=0}^{\min\{b_\Gamma, p\}} \binom{b_\Gamma}{i} (-1)^i \varphi_{1, \dim\widetilde{\Psi}(\Gamma) - p + i}(\widetilde{\Psi}(\Gamma)) \\ &= \sum_{\tau \prec \gamma''} (-1)^{\dim\tau + 1} \sum_{i=0}^{\min\{b_\tau, p\}} \binom{b_\tau}{i} (-1)^i \\ &\qquad \times \left\{ \varphi_{1, \dim\Psi(\tau) - p + i}(\Psi(\tau)) - \varphi_{1, \dim\widetilde{\Psi}(\square_\tau) - p + i}(\widetilde{\Psi}(\square_\tau)) \right\}, \end{aligned} \tag{7.17}$$

where the faces τ of the top γ' of $\square_{\gamma''}$ are neglected by the condition $\dim\widetilde{\Psi}(\tau) = 0$. By $\widetilde{\Psi}(\square_\tau) = \Delta_{\Psi(\tau)}$ and Lemma 7.13 below, the last term is equal to 0. □

Lemma 7.13 *For any face* γ *of* $\Gamma_+(f)$ *such that* $\gamma \subset \Gamma_f$*, we have*

$$\varphi_{1,j+1}(\Delta_\gamma) = \varphi_{1,j}(\gamma). \tag{7.18}$$

Proof. By the relation $l^*((k+1)\Delta_\gamma)_1 - l^*(k\Delta_\gamma)_1 = l^*(k\gamma)_1$ $(k \geq 0)$ we have

$$P_1(\Delta_\gamma; t) = tP_1(\gamma; t). \tag{7.19}$$

By comparing the coefficients of t^{j+1} in both sides, we obtain (7.18). □

The following proposition is a key in the proof of Proposition 7.11.

Proposition 7.14 *For any face γ of $\Gamma_+(f)$ such that $\gamma \subset \Gamma_f$, we have*

$$e(\chi_h([Z^*_{\Delta_\gamma}]+[Z^*_\gamma]))_1=(t_1t_2-1)^{\dim\gamma}. \tag{7.20}$$

Proof. It is enough to prove

$$e^{p,q}(Z^*_\gamma)_1+e^{p,q}(Z^*_{\Delta_\gamma})_1=(-1)^{\dim\gamma+p}\binom{\dim\gamma}{p}\cdot\delta_{p,q}, \tag{7.21}$$

where $\delta_{p,q}$ is Kronecker's delta. We consider the closure $\overline{Z^*_{\Delta_\gamma}}$ of $Z^*_{\Delta_\gamma}$ in $X_{\square_{\gamma''}}$. Then by the proofs of Propositions 2.12 and 7.12, we have

$$\begin{aligned} e^{p,q}(\overline{Z^*_{\Delta_\gamma}})_1 &= \sum_{\tau\prec\gamma''}\left\{e^{p,q}((\mathbb{C}^*)^{b_\tau}\times Z^*_{\Psi(\tau)})_1+e^{p,q}((\mathbb{C}^*)^{b_{\square_\tau}}\times Z^*_{\widetilde{\Psi}(\square_\tau)})_1\right\} && (7.22)\\ &= \sum_{\tau\prec\gamma''}\sum_{i=0}^{\min\{b_\tau,p\}}\binom{b_\tau}{i}(-1)^{i+b_\tau}\left\{e^{p-i,q-i}(Z^*_{\Psi(\tau)})_1+e^{p-i,q-i}(Z^*_{\Delta_{\Psi(\tau)}})_1\right\}. && (7.23)\end{aligned}$$

Let us prove (7.21) by induction on $\dim\gamma$. In the case $\dim\gamma=0$, we can prove (7.21) easily by Propositions 2.6 and 2.9. Assume that for any $\sigma\subset\Gamma_f$ such that $\dim\sigma<\dim\gamma$ (7.21) holds. Then by $b_{\gamma''}=0$ and (7.23) we have

$$e^{p,q}(\overline{Z^*_{\Delta_\gamma}})_1=e^{p,q}(Z^*_\gamma)_1+e^{p,q}(Z^*_{\Delta_\gamma})_1+\delta_{p,q}\sum_{\tau\precneqq\gamma''}(-1)^{\dim\tau+p}\binom{\dim\tau}{p}. \tag{7.24}$$

In the case $p+q>\dim\gamma$, by Proposition 2.6 we have

$$e^{p,q}(\overline{Z^*_{\Delta_\gamma}})_1=\delta_{p,q}\sum_{\tau\prec\gamma''}(-1)^{\dim\tau+p}\binom{\dim\tau}{p}. \tag{7.25}$$

Therefore, also in the case $p+q<\dim\gamma$, by the Poincaré duality for $\overline{Z^*_{\Delta_\gamma}}$ ($\square_{\gamma''}$ is prime) and Lemma 2.16 we have

$$\begin{aligned} e^{p,q}(\overline{Z^*_{\Delta_\gamma}})_1 &= e^{\dim\gamma-p,\dim\gamma-q}(\overline{Z^*_{\Delta_\gamma}})_1 && (7.26)\\ &= \delta_{p,q}\sum_{\tau\prec\gamma''}(-1)^{\dim\tau+\dim\gamma-p}\binom{\dim\tau}{\dim\gamma-p} && (7.27)\\ &= \delta_{p,q}\sum_{\tau\prec\gamma''}(-1)^{\dim\tau+p}\binom{\dim\tau}{p}. && (7.28)\end{aligned}$$

In the case $p+q=\dim\gamma$, by Proposition 7.12 and the previous results we have

$$\begin{aligned} e^{p,q}(\overline{Z^*_{\Delta_\gamma}})_1 &= \sum_{q'}e^{p,q'}(\overline{Z^*_{\Delta_\gamma}})_1-(1-\delta_{p,q})e^{p,p}(\overline{Z^*_{\Delta_\gamma}})_1 && (7.29)\\ &= \delta_{p,q}\sum_{\tau\prec\gamma''}(-1)^{\dim\tau+p}\binom{\dim\tau}{p}. && (7.30)\end{aligned}$$

By (7.24), we obtain (7.21) for any p,q. □

Now we can finish the proof of Proposition 7.11 as follows. By Proposition 7.14, we have

$$\sum_{\gamma\subset\Gamma_f} e\left(\chi_h\left((1-\mathbb{L})^{m_\gamma+1}([Z^*_{\Delta_\gamma}]+[Z^*_\gamma])\right)\right)_1 = \sum_{\gamma\subset\Gamma_f}(1-t_1t_2)^{m_\gamma+1}(t_1t_2-1)^{\dim\gamma} \tag{7.31}$$

$$= \sum_{l=1}^{n}(1-t_1t_2)^l \sum_{\sharp S_\gamma=l}(-1)^{\dim\gamma} \tag{7.32}$$

$$= \sum_{l=1}^{n}(1-t_1t_2)^l\binom{n}{l}(-1)^{l-1} \tag{7.33}$$

$$= 1-(t_1t_2)^n. \tag{7.34}$$

□

Remark 7.15 Following the proof of Theorem 5.16, we can easily give another proof to the Steenbrink conjecture which was proved by Varchenko-Khovanskii [50] and Saito [41] independently. For an introduction to this conjecture, see an excellent survey in Kulikov [22] etc.

Remark 7.16 For a bifurcation point $b\in B_f$ of f, take a small circle $C_\varepsilon(b)=\{x\in\mathbb{C}\mid |x-b|=\varepsilon\}$ $(0<\varepsilon\ll 1)$ around b such that $B_f\cap\{x\in\mathbb{C}\mid |x-b|\le\varepsilon\}=\{b\}$. Then by the restriction of $\mathbb{C}^n\setminus f^{-1}(B_f)\longrightarrow\mathbb{C}\setminus B_f$ to $C_\varepsilon(b)\subset\mathbb{C}\setminus B_f$ we obtain a geometric monodromy automorphism $\Phi^b_f\colon f^{-1}(b+\varepsilon)\xrightarrow{\sim} f^{-1}(b+\varepsilon)$ and the linear maps

$$\Phi^b_j\colon H^j(f^{-1}(b+\varepsilon);\mathbb{C})\xrightarrow{\sim} H^j(f^{-1}(b+\varepsilon);\mathbb{C})\quad (j=0,1,\ldots) \tag{7.35}$$

associated to it. The eigenvalues of Φ^b_j were studied in [28, Sections 3 and 4]. If f is tame at infinity, we can introduce a motivic Milnor fiber $\mathcal{S}^b_f\in\mathcal{M}^{\hat{\mu}}_{\mathbb{C}}$ along the central fiber $f^{-1}(b)$ to calculate the numbers of the Jordan blocks for the eigenvalues $\lambda\neq 1$ in Φ^b_{n-1}. This result can be easily obtained by the proof of [38, Theorem 13.1]. It would be an interesting problem to construct a motivic object to calculate the eigenvalue 1 part of Φ^b_{n-1}.

A Appendix by Claude Sabbah

In this appendix, we prove Theorems 4.5 and 4.6 of the main article.

A.1 Symmetry of Hodge numbers

Let U be a smooth affine complex variety and let $\mathscr{O}^{\mathrm{H}}_U$ denote the mixed Hodge module also denoted by $\mathbb{Q}^{\mathrm{H}}_U$ in [42] (we change the notation because we will mainly work with filtered $\mathscr{D}$-modules). We set $n=\dim U$ and $m=n-1$. Let $\mathbf{D}$ be the duality functor of algebraic mixed Hodge modules. We have $\mathbf{D}\mathscr{O}^{\mathrm{H}}_U\simeq\mathscr{O}^{\mathrm{H}}_U(n)$.

Let $f:U\to\mathbb{A}^1$ be a regular function. We denote by $f_*, f_!$ be the push-forward and proper push-forward functors (mainly used at the level of filtered $\mathscr{D}$-modules), where $f_!=\mathbf{D}f_*\mathbf{D}$ (cf. [42, (4.3.5)]).

Let t be the coordinate on $\mathbb{A}^1$. We will also use the nearby cycle functor $\psi_{1/t}$, that we decompose as $\psi_{1/t} = \psi_{1/f,1} \oplus \psi_{1/f,\neq 1}$ with respect to the eigenvalues of the monodromy. We have the following commutation relations in $\mathrm{MHM}(\mathbb{A}^1)$ (cf. [42, Prop. 2.6]):

$$\psi_{1/t}\mathbf{D} = (\mathbf{D}\psi_{1/t})(1).$$

According to the previous relations, we have

$$\psi_{1/t}(\mathscr{H}^0 f_! \mathscr{O}_U^{\mathrm{H}}) \simeq \psi_{1/t}(\mathbf{D}\mathscr{H}^0 f_* \mathbf{D}\mathscr{O}_U^{\mathrm{H}}) \simeq \mathbf{D}\big(\psi_{1/t}(\mathscr{H}^0 f_* \mathscr{O}_U^{\mathrm{H}})\big)(-m).$$

We denote by $h_!^{p,q}$ the Hodge numbers of the left-hand term, and by $h_*^{p,q}$ those of $\psi_{1/t}(\mathscr{H}^0 f_* \mathscr{O}_U^{\mathrm{H}})$. We then get

$$\forall p,q \in \mathbb{Z}, \qquad h_!^{p,q} = h_*^{m-p,m-q}, \tag{!$*$}$$

or equivalently, for each eigenvalue $\alpha \in \exp(2\pi i\mathbb{Q})$,

$$\forall p,q \in \mathbb{Z}, \qquad h_{!,\alpha}^{p,q} = h_{*,\alpha^{-1}}^{m-p,m-q} = h_{*,\alpha}^{m-q,m-p}, \tag{!$*$}$_\alpha$$

according to the behaviour of eigenvalues by duality and complex conjugation, if we note that $\alpha^{-1} = \overline{\alpha}$ for $\alpha \in \exp(2\pi i\mathbb{Q})$. From now on, we assume that f is cohomologically tame, in the sense of [38]. If $U = \mathbb{A}^n$ and $n \geqslant 2$, $\psi_{1/t,\neq 1}(\mathscr{H}^k f_! \mathscr{O}_U^{\mathrm{H}}) = 0$ for $k \neq 0$ (see [38, Rem. 10.3] for $\mathscr{H}^k f_*$ and use duality). Therefore, using the notation $e^{p,q}$ of (2.8) and (2.9) in the main part of the article, we have $e_{\neq 1}^{p,q} = h_{!,\neq 1}^{p,q}$, and we wish to show the symmetry $h_{!,\alpha}^{p,q} = h_{!,\alpha}^{m-p,m-q}$ for $\alpha \neq 1$. If $\alpha = 1$, the point is to show the symmetry $h_{!,1}^{p,q} = h_{!,1}^{m-1-p,m-1-q}$, since $f_! \mathscr{O}_U^{\mathrm{H}}$ has cohomology in degrees 0 and m at most and $\mathscr{H}^m$ has rank one. By $(!*)_\alpha$, these symmetries are equivalent to $h_{*,\alpha}^{p,q} = h_{*,\alpha}^{m-p,m-q}$ for $\alpha \neq 1$, and $h_{*,1}^{p,q} = h_{*,1}^{m+1-p,m+1-q}$. Both are a direct consequence of the following proposition, since $\widetilde{\mathrm{N}}$ is a morphism of type $(-1,-1)$.

Proposition A.1 *The weight filtration on $\psi_{1/t,1}(\mathscr{H}^0 f_* \mathscr{O}_U^{\mathrm{H}})$ (resp. on $\psi_{1/t,\neq 1}(\mathscr{H}^0 f_* \mathscr{O}_U^{\mathrm{H}})$) is equal to the monodromy filtration of the nilpotent part of the monodromy, centered at $m+1$ (resp. m).*

Notice also that Theorem 4.6 of the main article is a consequence of this statement. One can obtain the proposition as a consequence of Theorem 13.1 in [38], but we will propose another proof, which avoids the main results of [38] related to Fourier transform, Brieskorn lattices and spectrum at infinity.

Let us first treat the case $\alpha \neq 1$. Let $F : X \to \mathbb{C}$ be a compactification of f with no vanishing cycle for $\mathbb{Q}_U$ on $X \smallsetminus U$ (tameness), and let $\mathrm{IC}_X(\mathbb{Q}_U)$ be the intersection complex of X. It corresponds to a pure Hodge module $j_{!*}\mathscr{O}_U^{\mathrm{H}}$, according to M. Saito, and $\mathscr{H}^0 F_*(j_{!*}\mathscr{O}_U^{\mathrm{H}})$ is pure. Moreover, we have two morphisms in $\mathrm{MHM}(\mathbb{A}^1)$

$$\mathscr{H}^0 f_! \mathscr{O}_U^{\mathrm{H}} \longrightarrow \mathscr{H}^0 F_*(j_{!*}\mathscr{O}_U^{\mathrm{H}}) \longrightarrow \mathscr{H}^0 f_* \mathscr{O}_U^{\mathrm{H}}$$

and for each morphism, the kernel and cokernel (in $\mathrm{MHM}(\mathbb{A}^1)$) are constant mixed Hodge modules.

It follows that the computation of $\psi_{1/t,\neq 1}\mathscr{H}^0 f_* \mathscr{O}_U^{\mathrm{H}}$ or $\psi_{1/t,\neq 1}\mathscr{H}^0 f_! \mathscr{O}_U^{\mathrm{H}}$ (where t is the coordinate on $\mathbb{A}^1$) coincides with the computation of $\psi_{1/t,\neq 1}\mathscr{H}^0 F_*(j_{!*}\mathscr{O}_U^{\mathrm{H}})$. There, we can apply the properties of pure Hodge modules and get that the weight filtration is the monodromy filtration shifted by m, according to [40]. The case $\alpha = 1$ will occupy the next sections.

A.2 A preliminary result

Let H be a finite dimensional vector space equipped with a nilpotent endomorphism $\widetilde{\mathrm{N}}$. We denote by $\mathrm{M}(\widetilde{\mathrm{N}},H)_\bullet$ the monodromy filtration of $\widetilde{\mathrm{N}}$ on H (centered at 0), so that $\widetilde{\mathrm{N}}(\mathrm{M}(\widetilde{\mathrm{N}},H)_k) \subset \mathrm{M}(\widetilde{\mathrm{N}},H)_{k-2}$ for any $k \in \mathbb{Z}$ and, for any $\ell \in \mathbb{N}^*$, $\widetilde{\mathrm{N}}^\ell$ induces an isomorphism $\mathrm{gr}_\ell^{\mathrm{M}(\widetilde{\mathrm{N}},H)} H \xrightarrow{\sim} \mathrm{gr}_{-\ell}^{\mathrm{M}(\widetilde{\mathrm{N}},H)} H$.

The space $H/\mathrm{Im}\widetilde{\mathrm{N}}$ is naturally decomposed into primitive subspaces $\mathrm{P}_0(H,\widetilde{\mathrm{N}}) \oplus \cdots \oplus \mathrm{P}_\ell(H,\widetilde{\mathrm{N}}) \oplus \cdots$, and the filtration induced by $\mathrm{M}(\widetilde{\mathrm{N}},H)_\bullet$ on $H/\mathrm{Im}\widetilde{\mathrm{N}}$ is the filtration by the degree of the primitive part. The following is straightforward, by using the Jordan normal form for instance.

Lemma A.2 *Let $L_\bullet H$ be an increasing exhaustive filtration of H such that $L_{-2} = 0$ and $L_{-1}H = \mathrm{Im}\widetilde{\mathrm{N}}$. Then the following properties are equivalent:*

(a) $\mathrm{M}(\widetilde{\mathrm{N}},H)_\bullet$ *is equal to the monodromy filtration of* $\widetilde{\mathrm{N}}$ *relative to* $L_\bullet H$,

(b) *for* $k \geqslant 0$, $L_k H = \mathrm{M}(\widetilde{\mathrm{N}},H)_k + \mathrm{Im}\widetilde{\mathrm{N}}$.

A.3 Vanishing of hypercohomology

Let M be a regular holonomic $\mathscr{D}$-module on the affine line $\mathbb{A}^1$ with coordinate t. The following operation defines a new regular holonomic $\mathscr{D}$-module $\widetilde{M}$ such that the de Rham hypercohomology $\boldsymbol{H}^*(\mathbb{A}^1, \mathrm{DR}(\widetilde{M}))$ is zero. Note that, because we work with regular holonomic $\mathscr{D}$-modules, there is no difference between the algebraic and the analytic de Rham hypercohomologies. Working with $\mathbb{C}[t]\langle\partial_t\rangle$-modules, this amounts to asking that $\partial_t : \widetilde{M} \to \widetilde{M}$ is bijective. This can easily be realized by the following operation:

$$\widetilde{M} = \mathbb{C}[\partial_t, \partial_t^{-1}] \otimes_{\mathbb{C}[\partial_t]} M,$$

but this operation is not easily extended to mixed Hodge modules, which is our main purpose.

We will now consider the $\mathbb{C}[t]\langle\partial_t\rangle$-module $\mathbb{C}[\partial_t, \partial_t^{-1}]$ as a mixed Hodge module, and we will denote it $\mathbb{C}[\partial_t, \partial_t^{-1}]^{\mathrm{H}}$. It is constructed as follows. Firstly, as a $\mathbb{C}[t]\langle\partial_t\rangle$-module, we have a natural exact sequence

$$0 \longrightarrow \mathbb{C}[\partial_t] \longrightarrow \mathbb{C}[\partial_t, \partial_t^{-1}] \longrightarrow \mathbb{C}[t] \longrightarrow 0$$

by presenting $\mathbb{C}[\partial_t, \partial_t^{-1}]$ as $\mathbb{C}[t]\langle\partial_t\rangle/(t\partial_t + 1)$. Denoting by $j : (\mathbb{A}^1)^* = \mathbb{A}^1 \smallsetminus \{0\} \hookrightarrow \mathbb{A}^1$ the open inclusion and by $i : \{0\} \hookrightarrow \mathbb{A}^1$ the complementary closed inclusion, it corresponds to the mixed Hodge module $j_!\mathscr{O}^{\mathrm{H}}_{(\mathbb{A}^1)^*}$. The previous exact sequence is the weight exact sequence:

$$W_0(j_!\mathscr{O}^{\mathrm{H}}_{(\mathbb{A}^1)^*}) = i_*\mathbb{Q}^{\mathrm{H}}_0, \quad \mathrm{gr}_1^W(j_!\mathscr{O}^{\mathrm{H}}_{(\mathbb{A}^1)^*}) = \mathscr{O}^{\mathrm{H}}_{\mathbb{A}^1}.$$

(Recall that, in the theory of mixed Hodge modules, $\mathscr{O}^{\mathrm{H}}_{\mathbb{A}^1}$ has weight $\dim \mathbb{A}^1 = 1$).

If M is a regular holonomic $\mathscr{D}_{\mathbb{A}^1}$-module, we thus set

$$\widetilde{M} = \mathscr{H}^0 s_*(M \boxtimes j_!\mathscr{O}_{(\mathbb{A}^1)^*})$$

where $s : \mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$ is the sum function $(x,y) \mapsto x+y$ and the direct image is taken in the sense of $\mathscr{D}$-modules. Similarly, if M is a mixed Hodge module, we can regard the

previous definition within the frame of mixed Hodge modules and define $\widetilde{M}$ as a mixed Hodge module. We have a natural morphism $M \to \widetilde{M}$, whose kernel and cokernel are constant mixed Hodge modules.

Let us assume that M is a pure Hodge module on $\mathbb{A}^1$, of weight w. Then its image in $\widetilde{M}$ is also a pure of weight w, and we still denote it by M. In other words, we will assume that M has no constant submodule. Then we have an exact sequence in $\mathrm{MHM}(\mathbb{A}^1)$:

$$0 \longrightarrow M \longrightarrow \widetilde{M} \longrightarrow M'' \longrightarrow 0 \tag{A.1}$$

and M'' is constant and has weights $\geqslant w+1$.

A.4 Nearby cycles

The exact sequence (A.1) induces an exact sequence of mixed Hodge structures after taking nearby cycles at infinity:

$$0 \longrightarrow \psi_{1/t,1}M \longrightarrow \psi_{1/t,1}\widetilde{M} \longrightarrow \psi_{1/t,1}M'' \longrightarrow 0.$$

The weight filtration on $\psi_{1/t,1}M$ is the monodromy filtration of the nilpotent part N of the monodromy at infinity, centered at $w-1$, that we write $\mathrm{M}(\mathrm{N}, \psi_{1/t,1}M)[w-1]_\bullet$. The weight filtration $W_\bullet$ of $\psi_{1/t,1}\widetilde{M}$ is the monodromy filtration of $\widetilde{\mathrm{N}}$ on $\psi_{1/t,1}\widetilde{M}$ relative to the filtration $L_\bullet$ induced by $W_{\bullet+1}\widetilde{M}$. Lastly, M'' is constant, so $\mathrm{N}''=0$ on $\psi_{1/t,1}M''$ and the weight filtration $W_\bullet\psi_{1/t,1}M''$ is equal to $\psi_{1/t,1}W_{\bullet+1}M''$.

Proposition A.3 *Under these assumptions, the weight filtration $W_\bullet$ on $\psi_{1/t,1}\widetilde{M}$ is equal to the (absolute) monodromy filtration $\mathrm{M}(\widetilde{\mathrm{N}})[w]_\bullet$ of $\widetilde{\mathrm{N}}$ centered at w, and $L_\bullet$ is given by Lemma A.2, up to a shift by w.*

Proof. We will show that the filtration $L_\bullet\psi_{1/t,1}\widetilde{M}$ defined above satisfies the assumption of Lemma A.2 (up to a shift by w) and that $\mathrm{M}(\widetilde{\mathrm{N}})[w]_\bullet$ is the weight filtration of $\psi_{1/t,1}\widetilde{M}$. Therefore, the property A.2(a) will be fulfilled, and thus the filtration $L_\bullet$ satisfies A.2(b).

Let us first give some properties of the filtration $L_\bullet$. In the exact sequence (A.1), the weight filtration of $\widetilde{M}$ satisfies $W_{w-1}\widetilde{M}=0$, $W_w\widetilde{M}=M$ and M'' has weights $\geqslant w+1$. Each $W_{k+1}M''$ ($k \geqslant w$) is a constant Hodge module, which is completely determined by $W_k\boldsymbol{H}^{-1}(\mathbb{A}^1, \mathrm{DR}(M''))$ (where the $\mathscr{D}$-module convention is used for the de Rham complex, that is, $\mathrm{DR}(M'')$ has terms in degrees -1 and 0). Since $\widetilde{M}$ has no global hypercohomology, we have an isomorphism of mixed Hodge structures

$$\boldsymbol{H}^{-1}(\mathbb{A}^1, \mathrm{DR}M'') \xrightarrow{\sim} \boldsymbol{H}^0(\mathbb{A}^1, \mathrm{DR}M).$$

The mixed Hodge structure on $\boldsymbol{H}^0(\mathbb{A}^1, \mathrm{DR}M)$ is described as follows. It has weights $\geqslant w$. Let us denote by $\mathscr{M}_{\min}$ the minimal extension of M by the inclusion $j:\mathbb{A}^1 \hookrightarrow \mathbb{P}^1$ and by $\mathscr{M}$ the maximal extension j_*M. Then $\mathscr{M}_{\min}$ is a pure Hodge module of weight w on $\mathbb{P}^1$, and

$$\mathrm{gr}^W_w\boldsymbol{H}^0(\mathbb{A}^1, \mathrm{DR}M) = W_w\boldsymbol{H}^0(\mathbb{A}^1, \mathrm{DR}M) = \boldsymbol{H}^0(\mathbb{P}^1, \mathrm{DR}\mathscr{M}_{\min}) \subset \boldsymbol{H}^0(\mathbb{A}^1, \mathrm{DR}M).$$

The quotient Hodge structure $\boldsymbol{H}^0(\mathbb{A}^1, \mathrm{DR}M)/W_w\boldsymbol{H}^0(\mathbb{A}^1, \mathrm{DR}M)$ is identified with $\boldsymbol{H}^0(\mathbb{P}^1, \mathrm{DR}(\mathscr{M}/\mathscr{M}_{\min}))$. Note that $\mathscr{M}/\mathscr{M}_{\min}$ is supported at infinity, and is identified

with the direct image by the inclusion $\infty \hookrightarrow \mathbb{P}^1$ of $\phi_{1/t,1}(\mathscr{M}/\mathscr{M}_{\min})$. Moreover, $\phi_{1/t,1}\mathscr{M}_{\min}$ is identified with $\mathrm{Im}\mathrm{N} : \psi_{1/t,1}\mathscr{M}_{\min} \to \psi_{1/t,1}\mathscr{M}_{\min}(-1)$ (cf. [40, Lemme 5.1.4]) and, since $\psi_{1/t,1}\mathscr{M}_{\min} \to \psi_{1/t,1}\mathscr{M}$ and $\mathrm{var} : \phi_{1/t,1}\mathscr{M} \to \psi_{1/t,1}\mathscr{M}(-1)$ are isomorphisms compatible with N, we get an identification of $\phi_{1/t,1}(\mathscr{M}/\mathscr{M}_{\min})$ with $\mathrm{coker}\mathrm{N} : \psi_{1/t,1}M \to \psi_{1/t,1}M(-1)$. The graded pieces are thus given by

$$\mathrm{gr}^W_{w+k+1}\boldsymbol{H}^0(\mathbb{A}^1, \mathrm{DR}M) \xrightarrow{\sim} \mathrm{P}_k(\mathrm{N}, \psi_{1/t,1}M)(-1), \quad \forall k \geqslant 0.$$

(Recall that $\psi_{1/t,1}M$ is a mixed Hodge module having weight filtration given by $W_\bullet\psi_{1/t,1}M = \mathrm{M}(\mathrm{N})[w-1]_\bullet$; then $\psi_{1/t,1}M(-1)$ is a mixed Hodge module with $W_\bullet\psi_{1/t,1}M(-1) = \mathrm{M}(\mathrm{N})[w+1]_\bullet$.)

This computation gives the weight filtration on $\boldsymbol{H}^{-1}(\mathbb{A}^1, \mathrm{DR}M'')$, and thus on M'' since this is a constant Hodge module:

$$\begin{aligned} W_{w+1}M'' &= \boldsymbol{H}^0(\mathbb{P}^1, \mathrm{DR}\mathscr{M}_{\min}) \otimes_{\mathbb{C}} \mathbb{C}[t]^{\mathrm{H}}, \\ \mathrm{gr}^W_{w+k+2}M'' &\simeq \mathrm{P}_k(\mathrm{N}, \psi_{1/t,1}M)(-1) \otimes_{\mathbb{C}} \mathbb{C}[t]^{\mathrm{H}}, \quad \forall k \geqslant 0. \end{aligned}$$

As a consequence, we get

$$\begin{aligned} \mathrm{gr}^L_w\psi_{1/t,1}\widetilde{M} &= \mathrm{gr}^W_w\psi_{1/t,1}M'' \simeq \boldsymbol{H}^0(\mathbb{P}^1, \mathrm{DR}\mathscr{M}_{\min}), \\ \mathrm{gr}^L_{w+k+1}\psi_{1/t,1}\widetilde{M} &= \mathrm{gr}^W_{w+k+1}\psi_{1/t,1}M'' \simeq \mathrm{P}_k(\mathrm{N}, \psi_{1/t,1}M)(-1), \quad \forall k \geqslant 0. \end{aligned} \tag{A.2}$$

On the other hand,

$$L_{w-1}\psi_{1/t,1}\widetilde{M} = \mathrm{gr}^L_{w-1}\psi_{1/t,1}\widetilde{M} = \psi_{1/t,1}M.$$

Proof that $L_{w-1}\psi_{1/t,1}\widetilde{M} = \mathrm{Im}\widetilde{\mathrm{N}}$

Since $\mathrm{N}'' = 0$, we have $\mathrm{Im}\widetilde{\mathrm{N}} \subset \psi_{1/t,1}M = L_{w-1}\psi_{1/t,1}\widetilde{M}$. We will prove equality by an argument of Fourier transform. Recall that the Fourier transform ${}^F\!M$ of M is a $\mathbb{C}[\tau]\langle\partial_\tau\rangle$-module, through the correspondence $\mathbb{C}[\tau]\langle\partial_\tau\rangle \xrightarrow{\sim} \mathbb{C}[t]\langle\partial_t\rangle$, $\tau \mapsto \partial_t$, $\partial_\tau \mapsto -t$.

Lemma A.4 *There is a functorial isomorphism*

$$(\phi_{\tau,1}{}^F\!M, {}^F\mathrm{N}) \xrightarrow{\sim} (\psi_{1/t,1}M, \mathrm{N}) \tag{A.3}$$

for any regular holonomic $\mathbb{C}[t]\langle\partial_t\rangle$-module M.

Proof. This is "well-known". The proof of [37, Prop. 4.1(ivb)] can be adapted to $\mathscr{D}$-modules to show that a similar assertion holds on the product space $\mathbb{P}^1_t \times \mathbb{A}^1_\tau$ for the pull-back p^*M of M twisted by the exponential $\mathscr{D}$-module $\mathscr{E}^{-t\tau}$ (kernel of the Laplace transform). Applying direct image by the projection $q : \mathbb{P}^1_t \times \mathbb{A}^1_\tau \to \mathbb{A}^1_\tau$ and the compatibility of the functor $\phi_{\tau,1}$ with direct images (cf. e.g. [30]), we obtain (A.3), since ${}^F\!M$ can also be computed as $\mathscr{H}^0p_*(p^*M \otimes \mathscr{E}^{-t\tau})$. $\square$

Notice that ${}^F\widetilde{M}$ is the localization with respect to τ of ${}^F\!M$. Then the natural map $\psi_{1/t,1}M \to \psi_{1/t,1}\widetilde{M}$ is identified with the natural morphism (variation) $\phi_{\tau,1}{}^F\!M \to \psi_{\tau,1}{}^F\!M$ via the commutative diagram:

$$\begin{tikzcd} \psi_{1/t,1}M \arrow[d] & \phi_{\tau,1}{}^F\!M \arrow[l, "\sim"'] \arrow[r, "\mathrm{var}_\tau"] \arrow[d] & \psi_{\tau,1}{}^F\!M \arrow[d, "\wr"] \\ \psi_{1/t,1}\widetilde{M} & \phi_{\tau,1}{}^F\widetilde{M} \arrow[l, "\sim"'] \arrow[r, "\sim", "\mathrm{var}_\tau"'] & \psi_{\tau,1}{}^F\widetilde{M} \end{tikzcd}$$

The point is now that M is a semi-simple $\mathbb{C}[t]\langle\partial_t\rangle$-module, as it underlies a pure Hodge module. Moreover, the natural morphism from ${}^F\!M$ to its localization ${}^F\!\widetilde{M}$ is injective. Therefore, ${}^F\!M$ is a semi-simple $\mathbb{C}[\tau]\langle\partial_\tau\rangle$-module and has no submodule supported on $\tau=0$. Hence, the dual $\mathbb{C}[\tau]\langle\partial_\tau\rangle$-module satisfies the same properties, and therefore is included in its localization at $\tau=0$. As a consequence, ${}^F\!M$ is a minimal extension at $\tau=0$ (i.e., has no sub or quotient module supported at $\tau=0$), which implies that $\phi_{\tau,1}{}^F\!M \simeq \operatorname{Im}({}^F\mathrm{N}:\psi_{\tau,1}{}^F\!M\to\psi_{\tau,1}{}^F\!M)$ (cf. [40, Lemme 5.1.4]), and using the previous diagram, this is equivalent to $\phi_{\tau,1}{}^F\!M \simeq \operatorname{Im}({}^F\widetilde{\mathrm{N}}:\phi_{\tau,1}{}^F\!\widetilde{M}\to\phi_{\tau,1}{}^F\!\widetilde{M})$. Taking the inverse isomorphism (A.3) gives the assertion. □

Purity of $\mathrm{gr}_\ell^{\mathrm{M}(\widetilde{\mathrm{N}})}\psi_{1/t,1}\widetilde{M}$ for $\ell\neq 0$

According to [40, Lemme 5.1.12], var_τ is strictly compatible with the monodromy filtration after a shift by -1. Using the previous commutative diagram, we conclude that the same property holds for the inclusion $\psi_{1/t,1}M\to\psi_{1/t,1}\widetilde{M}$. On the other hand, $\mathrm{M}_{-1}(\widetilde{\mathrm{N}})\subset\operatorname{Im}\widetilde{\mathrm{N}}=\psi_{1/t,1}M$. Therefore, the previous inclusion induces isomorphisms

$$\mathrm{gr}_{\ell+1}^{\mathrm{M}(\mathrm{N})}\psi_{1/t,1}M \xrightarrow{\sim} \mathrm{gr}_\ell^{\mathrm{M}(\widetilde{\mathrm{N}})}\psi_{1/t,1}\widetilde{M} \tag{A.4}$$

for each $\ell\leqslant -1$. Remark now that such morphisms underly morphisms of mixed Hodge structures, since $\mathrm{M}_\bullet$ is a filtration by mixed Hodge structures. By strictness, the corresponding morphisms of mixed Hodge structures are isomorphisms. Since the left-hand term is pure of weight $w+\ell$, the right-hand term is so. Lastly, since $\widetilde{\mathrm{N}}^\ell:\mathrm{gr}_\ell^{\mathrm{M}(\widetilde{\mathrm{N}})}\psi_{1/t,1}\widetilde{M}\xrightarrow{\sim}\mathrm{gr}_{-\ell}^{\mathrm{M}(\widetilde{\mathrm{N}})}\psi_{1/t,1}\widetilde{M}(-\ell)$ is an isomorphism of mixed Hodge structures, we conclude that $\mathrm{gr}_\ell^{\mathrm{M}(\widetilde{\mathrm{N}})}\psi_{1/t,1}\widetilde{M}$ is pure of weight $w+\ell$ for $\ell\geqslant 1$. □

Dimension of $\mathrm{gr}^W_{w+\ell}\psi_{1/t,1}\widetilde{M}$

We now consider the weight filtration $W_\bullet\psi_{1/t,1}\widetilde{M}$ of the mixed Hodge structure $\psi_{1/t,1}\widetilde{M}$. We claim that

$$\forall\ell,\quad \dim\mathrm{gr}^W_{w+\ell}\psi_{1/t,1}\widetilde{M}=\dim\mathrm{gr}_\ell^{\mathrm{M}(\widetilde{\mathrm{N}})}\psi_{1/t,1}\widetilde{M}. \tag{A.5}$$

Notice that, since both filtrations are exhaustive, it is enough to prove the claim for $\ell\neq 0$. Assume first that $\ell\leqslant -1$. On the one hand, we have by (A.4)

$$\dim\mathrm{gr}_\ell^{\mathrm{M}(\widetilde{\mathrm{N}})}\psi_{1/t,1}\widetilde{M}=\dim\mathrm{gr}_{\ell+1}^{\mathrm{M}(\mathrm{N})}\psi_{1/t,1}M=\dim\mathrm{gr}^W_{w+\ell}\psi_{1/t,1}M.$$

On the other hand, since $W_\bullet\psi_{1/t,1}M''=L_\bullet\psi_{1/t,1}M''$ and $L_{w+\ell}\psi_{1/t,1}M''=0$ for $\ell\leqslant -1$, the natural morphism

$$W_{w+\ell}\psi_{1/t,1}M\longrightarrow W_{w+\ell}\psi_{1/t,1}\widetilde{M} \tag{A.6}$$

is an isomorphism, hence the assertion for $\ell\leqslant -1$. Assume now that $\ell\geqslant 1$. We have

$$\begin{aligned}\dim\mathrm{gr}^W_{w+\ell}\psi_{1/t,1}\widetilde{M}&=\dim\mathrm{gr}^W_{w+\ell}\psi_{1/t,1}M+\dim\mathrm{gr}^W_{w+\ell}\psi_{1/t,1}M''\\&=\dim\mathrm{gr}_{\ell+1}^{\mathrm{M}(\mathrm{N})}\psi_{1/t,1}M+\dim\mathrm{P}\mathrm{gr}_{\ell-1}^{\mathrm{M}(\mathrm{N})}\psi_{1/t,1}M\\&=\dim\mathrm{gr}_{\ell-1}^{\mathrm{M}(\mathrm{N})}\psi_{1/t,1}M.\end{aligned}$$

On the other hand,

$$\begin{aligned}\dim \mathrm{gr}_{\ell}^{\mathrm{M}(\widetilde{\mathrm{N}})}\psi_{1/t,1}\widetilde{M} &= \dim \mathrm{gr}_{-\ell}^{\mathrm{M}(\widetilde{\mathrm{N}})}\psi_{1/t,1}\widetilde{M} \\ &= \dim \mathrm{gr}_{-\ell+1}^{\mathrm{M}(\mathrm{N})}\psi_{1/t,1}M = \dim \mathrm{gr}_{\ell-1}^{\mathrm{M}(\mathrm{N})}\psi_{1/t,1}M,\end{aligned}$$

so (A.5) is proved. □

End of the proof of Proposition A.3

The purity of $\mathrm{gr}_{\ell}^{\mathrm{M}(\widetilde{\mathrm{N}})}\psi_{1/t,1}\widetilde{M}$ shows that $W_\bullet$ induces the trivial filtration with one jump from $w+\ell-1$ to $w+\ell$ on $\mathrm{gr}_{\ell}^{\mathrm{M}(\widetilde{\mathrm{N}})}\psi_{1/t,1}\widetilde{M}$, for $\ell \neq 0$. In particular, for $\ell \neq 0$,

$$W_{w+\ell-1} \cap \mathrm{M}(\widetilde{\mathrm{N}})_\ell \subset \mathrm{M}(\widetilde{\mathrm{N}})_{\ell-1}. \tag{A.7}$$

Let $\ell_o \gg 0$ be such that $W_{w+\ell_o} = \mathrm{M}(\widetilde{\mathrm{N}})_{\ell_o} = \psi_{1/t,1}\widetilde{M}$. Then (A.7) shows that $W_{w+\ell_o-1} \subset \mathrm{M}(\widetilde{\mathrm{N}})_{\ell_o-1}$, and (A.5) for $\ell = \ell_o$ implies equality. A similar argument can be applied by decreasing induction up to $\ell = 1$, giving $W_{w+\ell} = \mathrm{M}(\widetilde{\mathrm{N}})_\ell$ for any $\ell \geqslant 0$. Assume now that $\ell \leqslant -1$. Then (A.6) shows that $W_{w+\ell} = \mathrm{M}(\mathrm{N}_{|\mathrm{Im}\widetilde{\mathrm{N}}})_{\ell+1}$. It is easy to check that this is nothing but $\mathrm{M}(\widetilde{\mathrm{N}})_\ell$. □

A.5 End of the proof of Proposition A.1

Recall that we assume that $U = \mathbb{C}^n$. Let us first show that, if we set $M = \mathscr{H}^0 F_*(j_{!*}\mathscr{O}_U^{\mathrm{H}})$, which is a pure Hodge module of weight n, according to M. Saito [40], we have $\mathscr{H}^0 f_*\mathscr{O}_U^{\mathrm{H}} \simeq \widetilde{M}$ as a mixed Hodge module. Indeed, by functoriality of the $\widetilde{\ }$ operation, we have commutative diagram in $\mathrm{MHM}(\mathbb{A}^1_t)$:

$$\begin{array}{ccc} M & \xrightarrow{a} & \mathscr{H}^0 f_*\mathscr{O}_U^{\mathrm{H}} \\ \downarrow & & \downarrow{\scriptstyle b} \\ \widetilde{M} & \xrightarrow{\widetilde{a}} & \widetilde{\mathscr{H}^0 f_*\mathscr{O}_U^{\mathrm{H}}} \end{array}$$

Since the kernel and the cokernel of a are constant, $\widetilde{a}$ is an isomorphism, and since the operator ∂_t is invertible on $\mathscr{H}^0 f_*\mathscr{O}_U$, b is an isomorphism. As a consequence, Proposition A.3 applies to $\mathscr{H}^0 f_*\mathscr{O}_U^{\mathrm{H}}$. □

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